Chapter 8

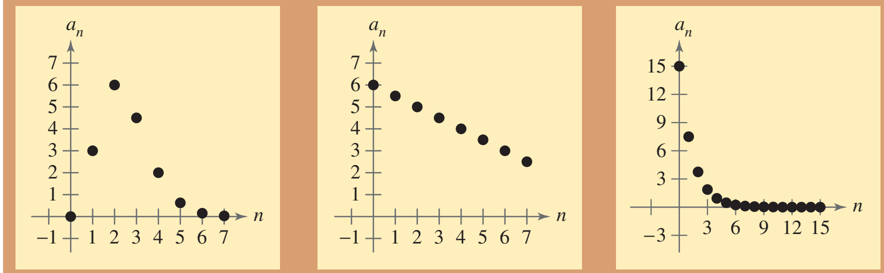
Sequences, Series, and Probability

8.1 Sequences and Series
8.2 Arithmetic Sequences and Partial Sums
8.3 Geometric Sequences and Series
8.4 Mathematical Induction
8.5 The Binomial Theorem
8.6 Counting Principles
8.7 Probability

Selected Applications

Sequences, series, and probability have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

- Average Wages, Exercise 119, page 590
- Falling Object, Exercise 84, page 599
- Multiplier Effect, Exercises 91–96, page 609
- Tower of Hanoi, Exercise 56, page 618
- Health, Exercise 109, page 626
- PIN Codes, Exercise 18, page 634
- Data Analysis, Exercise 35, page 647
- Course Schedule, Exercise 113, page 653



Sequences and series describe algebraic patterns. Graphs of sequences allow you to obtain a graphical perspective of the algebraic pattern described. In Chapter 8, you will study sequences and series extensively. You will also learn how to use mathematical induction to prove formulas and how to use the Binomial Theorem to calculate binomial coefficients, and you will study probability theory.

Bill Lai/Index Stock

Personal identification numbers, or PINs, are numerical passcodes for accessing such things as automatic teller machines, online accounts, and entrance to secure buildings. PINs are randomly generated, or consumers can create their own PIN.

8.1 Sequences and Series

What you should learn

- Use sequence notation to write the terms of sequences.
- Use factorial notation.
- Use summation notation to write sums.
- Find sums of infinite series.
- Use sequences and series to model and solve real-life problems.

Why you should learn it

Sequences and series are useful in modeling sets of values in order to identify patterns. For instance, Exercise 121 on page 590 shows how a sequence can be used to model the revenue of a pizza franchise from 1999 to 2006.

Santi Visalli/Tips Images

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English. Saying that a collection is listed *in sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers. Instead of using function notation, sequences are usually written using subscript notation, as shown in the following definition.

> **Definition of Sequence**
>
> An **infinite sequence** is a function whose domain is the set of positive integers. The function values
>
> $$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$
>
> are the **terms** of the sequence. If the domain of a function consists of the first n positive integers only, the sequence is a **finite sequence.**

On occasion, it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become $a_0, a_1, a_2, a_3, \ldots$. The domain of the function is the set of nonnegative integers.

Example 1 Writing the Terms of a Sequence

Write the first four terms of each sequence.

a. $a_n = 3n - 2$ **b.** $a_n = 3 + (-1)^n$

Solution

a. The first four terms of the sequence given by $a_n = 3n - 2$ are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10.$ 4th term

b. The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$a_1 = 3 + (-1)^1 = 3 - 1 = 2$ 1st term

$a_2 = 3 + (-1)^2 = 3 + 1 = 4$ 2nd term

$a_3 = 3 + (-1)^3 = 3 - 1 = 2$ 3rd term

$a_4 = 3 + (-1)^4 = 3 + 1 = 4.$ 4th term

CHECKPOINT Now try Exercise 1.

TECHNOLOGY TIP

To graph a sequence using a graphing utility, set the mode to *dot* and *sequence* and enter the sequence. Try graphing the sequences in Example 1 and using the *value* or *trace* feature to identify the terms. For instructions on how to use the *dot* mode, *sequence* mode, *value* feature, and *trace* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Example 2 Writing the Terms of a Sequence

Write the first five terms of the sequence given by $a_n = \dfrac{(-1)^n}{2n - 1}$.

Algebraic Solution

The first five terms of the sequence are as follows.

$$a_1 = \frac{(-1)^1}{2(1) - 1} = \frac{-1}{2 - 1} = -1 \qquad \text{1st term}$$

$$a_2 = \frac{(-1)^2}{2(2) - 1} = \frac{1}{4 - 1} = \frac{1}{3} \qquad \text{2nd term}$$

$$a_3 = \frac{(-1)^3}{2(3) - 1} = \frac{-1}{6 - 1} = -\frac{1}{5} \qquad \text{3rd term}$$

$$a_4 = \frac{(-1)^4}{2(4) - 1} = \frac{1}{8 - 1} = \frac{1}{7} \qquad \text{4th term}$$

$$a_5 = \frac{(-1)^5}{2(5) - 1} = \frac{-1}{10 - 1} = -\frac{1}{9} \qquad \text{5th term}$$

CHECKPOINT Now try Exercise 11.

Numerical Solution

Set your graphing utility to *sequence* mode. Enter the sequence into your graphing utility, as shown in Figure 8.1. Use the *table* feature (in *ask* mode) to create a table showing the terms of the sequence u_n for $n = 1, 2, 3, 4$, and 5. From Figure 8.2, you can estimate the first five terms of the sequence as follows.

$$u_1 = -1, \qquad u_2 = 0.33333 \approx \tfrac{1}{3}, \qquad u_3 = -0.2 = -\tfrac{1}{5},$$

$$u_4 = 0.14286 \approx \tfrac{1}{7}, \qquad \text{and} \qquad u_5 = -0.1111 \approx -\tfrac{1}{9}$$

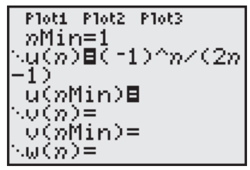


Figure 8.1

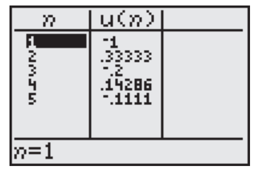


Figure 8.2

Simply listing the first few terms is not sufficient to define a unique sequence—the nth term *must be given*. To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n + 1)(n^2 - n + 6)}, \ldots$$

TECHNOLOGY SUPPORT

For instructions on how to use the *table* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center.*

Example 3 Finding the nth Term of a Sequence

Write an expression for the apparent nth term (a_n) of each sequence.

a. 1, 3, 5, 7, . . . **b.** 2, 5, 10, 17, . . .

Solution

a. n: 1 2 3 4 . . . n

Terms: 1 3 5 7 . . . a_n

Apparent Pattern: Each term is 1 less than twice n. So, the apparent nth term is $a_n = 2n - 1$.

b. n: 1 2 3 4 . . . n

Terms: 2 5 10 17 . . . a_n

Apparent Pattern: Each term is 1 more than the square of n. So, the apparent nth term is $a_n = n^2 + 1$.

CHECKPOINT Now try Exercise 43.

Some sequences are defined **recursively.** To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms. A well-known example is the Fibonacci sequence, shown in Example 4.

Example 4 The Fibonacci Sequence: A Recursive Sequence

The Fibonacci sequence is defined recursively as follows.

$$a_0 = 1,\ a_1 = 1,\ a_k = a_{k-2} + a_{k-1}, \qquad \text{where } k \geq 2$$

Write the first six terms of this sequence.

Solution

$a_0 = 1$ — 0th term is given.

$a_1 = 1$ — 1st term is given.

$a_2 = a_{2-2} + a_{2-1} = a_0 + a_1 = 1 + 1 = 2$ — Use recursion formula.

$a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 1 + 2 = 3$ — Use recursion formula.

$a_4 = a_{4-2} + a_{4-1} = a_2 + a_3 = 2 + 3 = 5$ — Use recursion formula.

$a_5 = a_{5-2} + a_{5-1} = a_3 + a_4 = 3 + 5 = 8$ — Use recursion formula.

CHECKPOINT Now try Exercise 57.

Factorial Notation

Some very important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

Definition of Factorial

If n is a positive integer, ***n* factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n-1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

Exploration

Most graphing utilities have the capability to compute $n!$. Use your graphing utility to compare $3 \cdot 5!$ and $(3 \cdot 5)!$. How do they differ? How large a value of $n!$ will your graphing utility allow you to compute?

Here are some values of $n!$ for the first few nonnegative integers. Notice that $0! = 1$ by definition.

$0! = 1$

$1! = 1$

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

The value of n does not have to be very large before the value of $n!$ becomes huge. For instance, $10! = 3{,}628{,}800$.

Factorials follow the same conventions for order of operations as do exponents. For instance,

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot n)$$

whereas $(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot 2n$.

Example 5 Writing the Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by $a_n = \frac{2^n}{n!}$. Begin with $n = 0$.

Algebraic Solution

$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1$ 0th term

$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2$ 1st term

$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2$ 2nd term

$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3}$ 3rd term

$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3}$ 4th term

CHECKPOINT Now try Exercise 65.

Graphical Solution

Using a graphing utility set to *dot* and *sequence* modes, enter the sequence $u_n = 2^n/n!$, as shown in Figure 8.3. Set the viewing window to $0 \le n \le 4$, $0 \le x \le 6$, and $0 \le y \le 4$. Then graph the sequence, as shown in Figure 8.4. Use the *value* or *trace* feature to approximate the first five terms as follows.

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 2, \quad u_3 \approx 1.333 \approx \tfrac{4}{3}, \quad u_4 \approx 0.667 \approx \tfrac{2}{3}$$

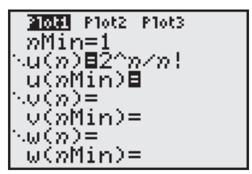


Figure 8.3

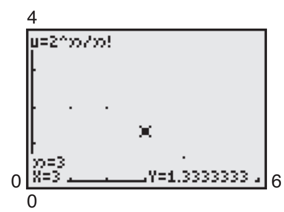


Figure 8.4

When working with fractions involving factorials, you will often find that the fractions can be reduced to simplify the computations.

Example 6 Evaluating Factorial Expressions

Simplify each factorial expression.

a. $\frac{8!}{2! \cdot 6!}$ **b.** $\frac{2! \cdot 6!}{3! \cdot 5!}$ **c.** $\frac{n!}{(n-1)!}$

Solution

a. $\frac{8!}{2! \cdot 6!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 7 \cdot 8}{1 \cdot 2 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}} = \frac{7 \cdot 8}{2} = 28$

b. $\frac{2! \cdot 6!}{3! \cdot 5!} = \frac{\cancel{1 \cdot 2} \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 6}{\cancel{1 \cdot 2} \cdot 3 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}} = \frac{6}{3} = 2$

c. $\frac{n!}{(n-1)!} = \frac{\cancel{1 \cdot 2 \cdot 3 \cdots (n-1)} \cdot n}{\cancel{1 \cdot 2 \cdot 3 \cdots (n-1)}} = n$

CHECKPOINT Now try Exercise 75.

STUDY TIP

Note in Example 6(a) that you can also simplify the computation as follows.

$$\frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7 \cdot \cancel{6!}}{2! \cdot \cancel{6!}} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

Summation Notation

There is a convenient notation for the sum of the terms of a finite sequence. It is called **summation notation** or **sigma notation** because it involves the use of the uppercase Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdot \cdot \cdot + a_n$$

where i is called the **index of summation,** n is the **upper limit of summation,** and 1 is the **lower limit of summation.**

> **STUDY TIP**
>
> Summation notation is an instruction to add the terms of a sequence. From the definition at the left, the upper limit of summation tells you where to end the sum. Summation notation helps you generate the appropriate terms of the sequence prior to finding the actual sum, which may be unclear.

Example 7 Sigma Notation for Sums

a. $$\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$$
$$= 3(1 + 2 + 3 + 4 + 5)$$
$$= 3(15)$$
$$= 45$$

b. $$\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$$
$$= 10 + 17 + 26 + 37 = 90$$

c. $$\sum_{n=0}^{8} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$$
$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}$$
$$\approx 2.71828$$

For the summation in part (c), note that the sum is very close to the irrational number $e \approx 2.718281828$. It can be shown that as more terms of the sequence whose nth term is $1/n!$ are added, the sum becomes closer and closer to e.

CHECKPOINT Now try Exercise 79.

> **TECHNOLOGY SUPPORT**
>
> For instructions on how to use the *sum sequence* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center.*

In Example 7, note that the lower limit of a summation does not have to be 1. Also note that the index of summation does not have to be the letter i. For instance, in part (b), the letter k is the index of summation.

TECHNOLOGY TIP Most graphing utilities are able to sum the first n terms of a sequence. Figure 8.5 shows an example of how one graphing utility displays the sum of the terms of the sequence below using the *sum sequence* feature.

$$a_n = \frac{1}{n!} \quad \text{from} \quad n = 0 \quad \text{to} \quad n = 8$$

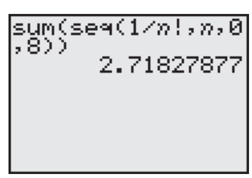


Figure 8.5

Properties of Sums (See the proofs on page 656.)

1. $\sum_{i=1}^{n} c = cn$, c is a constant.
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, c is a constant.
3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

STUDY TIP

Variations in the upper and lower limits of summation can produce quite different-looking summation notations for *the same sum*. For example, the following two sums have identical terms.

$$\sum_{i=1}^{3} 3(2^i) = 3(2^1 + 2^2 + 2^3)$$

$$\sum_{i=0}^{2} 3(2^{i+1}) = 3(2^1 + 2^2 + 2^3)$$

Series

Many applications involve the sum of the terms of a finite or an infinite sequence. Such a sum is called a **series.**

Definition of a Series

Consider the infinite sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$.

1. The sum of the first n terms of the sequence is called a **finite series** or the **partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

2. The sum of all the terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

Example 8 Finding the Sum of a Series

For the series $\sum_{i=1}^{\infty} \frac{3}{10^i}$, find (a) the third partial sum and (b) the sum.

Solution

a. The third partial sum is

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} = 0.3 + 0.03 + 0.003 = 0.333.$$

b. The sum of the series is

$$\sum_{i=1}^{\infty} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots$$

$$= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots$$

$$= 0.33333\ldots = \frac{1}{3}.$$

CHECKPOINT Now try Exercise 109.

Notice in Example 8(b) that the sum of an infinite series can be a finite number.

Application

Sequences have many applications in situations that involve recognizable patterns. One such model is illustrated in Example 9.

Example 9 Population of the United States

From 1970 to 2004, the resident population of the United States can be approximated by the model

$$a_n = 205.5 + 1.82n + 0.024n^2, \qquad n = 0, 1, \ldots, 34$$

where a_n is the population (in millions) and n represents the year, with $n = 0$ corresponding to 1970. Find the last five terms of this finite sequence. (Source: U.S. Census Bureau)

Algebraic Solution

The last five terms of this finite sequence are as follows.

$$a_{30} = 205.5 + 1.82(30) + 0.024(30)^2$$

$$= 281.7 \qquad \text{2000 population}$$

$$a_{31} = 205.5 + 1.82(31) + 0.024(31)^2$$

$$\approx 285.0 \qquad \text{2001 population}$$

$$a_{32} = 205.5 + 1.82(32) + 0.024(32)^2$$

$$\approx 288.3 \qquad \text{2002 population}$$

$$a_{33} = 205.5 + 1.82(33) + 0.024(33)^2$$

$$\approx 291.7 \qquad \text{2003 population}$$

$$a_{34} = 205.5 + 1.82(34) + 0.024(34)^2$$

$$\approx 295.1 \qquad \text{2004 population}$$

CHECKPOINT Now try Exercise 113.

Graphical Solution

Using a graphing utility set to *dot* and *sequence* modes, enter the sequence

$$u_n = 205.5 + 1.82n + 0.024n^2.$$

Set the viewing window to $0 \le n \le 35$, $0 \le x \le 35$, and $200 \le y \le 300$. Then graph the sequence. Use the *value* or *trace* feature to approximate the last five terms, as shown in Figure 8.6.

$$a_{30} = 281.7,$$
$$a_{31} \approx 285.0,$$
$$a_{32} \approx 288.3,$$
$$a_{33} \approx 291.7,$$
$$a_{34} \approx 295.1$$

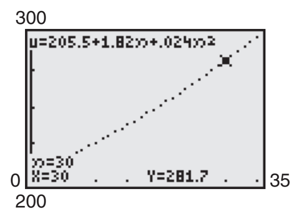


Figure 8.6

Exploration

A 3 × 3 × 3 cube is created using 27 unit cubes (a unit cube has a length, width, and height of 1 unit) and only the faces of each cube that are visible are painted blue (see Figure 8.7). Complete the table below to determine how many unit cubes of the 3 × 3 × 3 cube have no blue faces, one blue face, two blue faces, and three blue faces. Do the same for a 4 × 4 × 4 cube, a 5 × 5 × 5 cube, and a 6 × 6 × 6 cube, and add your results to the table below. What type of pattern do you observe in the table? Write a formula you could use to determine the column values for an $n \times n \times n$ cube.

Cube \ Number of blue faces	0	1	2	3
3 × 3 × 3				

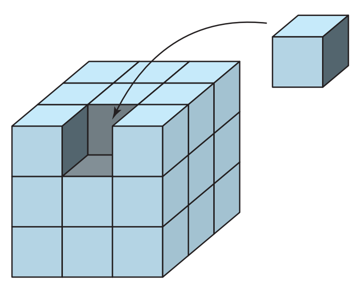


Figure 8.7

8.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. An _______ is a function whose domain is the set of positive integers.

2. The function values $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ are called the _______ of a sequence.

3. A sequence is a ________ sequence if the domain of the function consists of the first n positive integers.

4. If you are given one or more of the first few terms of a sequence, and all other terms of the sequence are defined using previous terms, then the sequence is defined _______ .

5. If n is a positive integer, n _______ is defined as $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$.

6. The notation used to represent the sum of the terms of a finite sequence is _______ or sigma notation.

7. For the sum $\sum_{i=1}^{n} a_i$, i is called the _______ of summation, n is the _______ of summation, and 1 is the _______ of summation.

8. The sum of the terms of a finite or an infinite sequence is called a ________

9. The _______ of a sequence is the sum of the first n terms of the sequence.

In Exercises 1–20, write the first five terms of the sequence. (Assume *n* begins with 1.) Use the *table* feature of a graphing utility to verify your results.

1. $a_n = 2n + 5$

2. $a_n = 4n - 7$

3. $a_n = 2^n$

4. $a_n = \left(\frac{1}{2}\right)^n$

5. $a_n = \left(-\frac{1}{2}\right)^n$

6. $a_n = (-2)^n$

7. $a_n = \dfrac{n+1}{n}$

8. $a_n = \dfrac{n}{n+1}$

9. $a_n = \dfrac{n}{n^2+1}$

10. $a_n = \dfrac{2n}{n+1}$

11. $a_n = \dfrac{1 + (-1)^n}{n}$

12. $a_n = \dfrac{1 + (-1)^n}{2n}$

13. $a_n = 1 - \dfrac{1}{2^n}$

14. $a_n = \dfrac{3^n}{4^n}$

15. $a_n = \dfrac{1}{n^{3/2}}$

16. $a_n = \dfrac{1}{\sqrt{n}}$

17. $a_n = \dfrac{(-1)^n}{n^2}$

18. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$

19. $a_n = (2n-1)(2n+1)$

20. $a_n = n(n-1)(n-2)$

In Exercises 21–26, find the indicated term of the sequence.

21. $a_n = (-1)^n(3n-2)$
$a_{25} = \square$

22. $a_n = (-1)^{n-1}[n(n-1)]$
$a_{16} = \square$

23. $a_n = \dfrac{n^2}{n^2+1}$
$a_{10} = \square$

24. $a_n = \dfrac{n^2}{2n+1}$
$a_5 = \square$

25. $a_n = \dfrac{2^n}{2^n+1}$
$a_6 = \square$

26. $a_n = \dfrac{2^{n+1}}{2^n+1}$
$a_7 = \square$

In Exercises 27–32, use a graphing utility to graph the first 10 terms of the sequence. (Assume *n* begins with 1.)

27. $a_n = \dfrac{2}{3}n$

28. $a_n = 2 - \dfrac{4}{n}$

29. $a_n = 16(-0.5)^{n-1}$

30. $a_n = 8(0.75)^{n-1}$

31. $a_n = \dfrac{2n}{n+1}$

32. $a_n = \dfrac{3n^2}{n^2+1}$

In Exercises 33–38, use the *table* feature of a graphing utility to find the first 10 terms of the sequence. (Assume *n* begins with 1.)

33. $a_n = 2(3n-1) + 5$

34. $a_n = 2n(n+1)(n+2)$

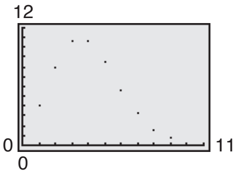
35. $a_n = 1 + \dfrac{n+1}{n}$

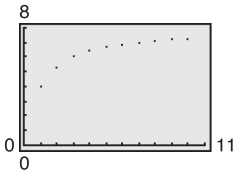
36. $a_n = \dfrac{4n^2}{n+2}$

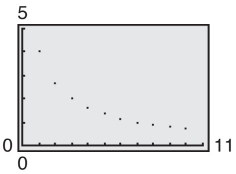
37. $a_n = (-1)^n + 1$

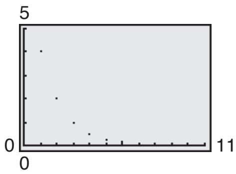
38. $a_n = (-1)^{n+1} + 1$

In Exercises 39–42, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)


(b)


(c)


(d)


39. $a_n = \dfrac{8}{n+1}$ **40.** $a_n = \dfrac{8n}{n+1}$

41. $a_n = 4(0.5)^{n-1}$ **42.** $a_n = \dfrac{4^n}{n!}$

In Exercises 43–56, write an expression for the *apparent n*th term of the sequence. (Assume *n* begins with 1.)

43. 1, 4, 7, 10, 13, . . . **44.** 3, 7, 11, 15, 19, . . .

45. 0, 3, 8, 15, 24, . . . **46.** $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

47. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$ **48.** $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

49. $\frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \ldots$ **50.** $\frac{1}{3}, -\frac{2}{9}, \frac{4}{27}, -\frac{8}{81}, \ldots$

51. $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

52. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

53. $1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots$ **54.** $1, 2, \dfrac{2^2}{2}, \dfrac{2^3}{6}, \dfrac{2^4}{24}, \dfrac{2^5}{120}, \ldots$

55. 1, 3, 1, 3, 1, . . . **56.** $1, -1, 1, -1, 1, \ldots$

In Exercises 57–60, write the first five terms of the sequence defined recursively.

57. $a_1 = 28, \quad a_{k+1} = a_k - 4$

58. $a_1 = 15, \quad a_{k+1} = a_k + 3$

59. $a_1 = 3, \quad a_{k+1} = 2(a_k - 1)$

60. $a_1 = 32, \quad a_{k+1} = \frac{1}{2}a_k$

In Exercises 61–64, write the first five terms of the sequence defined recursively. Use the pattern to write the *n*th term of the sequence as a function of *n*. (Assume *n* begins with 1.)

61. $a_1 = 6, \quad a_{k+1} = a_k + 2$

62. $a_1 = 25, \quad a_{k+1} = a_k - 5$

63. $a_1 = 81, \quad a_{k+1} = \frac{1}{3}a_k$

64. $a_1 = 14, \quad a_{k+1} = -2a_k$

In Exercises 65–70, write the first five terms of the sequence. (Assume *n* begins with 0.) Use the *table* feature of a graphing utility to verify your results.

65. $a_n = \dfrac{1}{n!}$ **66.** $a_n = \dfrac{1}{(n+1)!}$

67. $a_n = \dfrac{n!}{2n+1}$ **68.** $a_n = \dfrac{n^2}{(n+1)!}$

69. $a_n = \dfrac{(-1)^{2n}}{(2n)!}$ **70.** $a_n = \dfrac{(-1)^{2n+1}}{(2n+1)!}$

In Exercises 71–78, simplify the factorial expression.

71. $\dfrac{2!}{4!}$ **72.** $\dfrac{5!}{7!}$

73. $\dfrac{12!}{4! \cdot 8!}$ **74.** $\dfrac{10! \cdot 3!}{4! \cdot 6!}$

75. $\dfrac{(n+1)!}{n!}$ **76.** $\dfrac{(n+2)!}{n!}$

77. $\dfrac{(2n-1)!}{(2n+1)!}$ **78.** $\dfrac{(2n+2)!}{(2n)!}$

In Exercises 79–90, find the sum.

79. $\displaystyle\sum_{i=1}^{5} (2i + 1)$ **80.** $\displaystyle\sum_{i=1}^{6} (3i - 1)$

81. $\displaystyle\sum_{k=1}^{4} 10$ **82.** $\displaystyle\sum_{k=1}^{5} 6$

83. $\displaystyle\sum_{i=0}^{4} i^2$ **84.** $\displaystyle\sum_{i=0}^{5} 3i^2$

85. $\displaystyle\sum_{k=0}^{3} \frac{1}{k^2+1}$ **86.** $\displaystyle\sum_{j=3}^{5} \frac{1}{j}$

87. $\displaystyle\sum_{i=1}^{4} [(i-1)^2 + (i+1)^3]$ **88.** $\displaystyle\sum_{k=2}^{5} (k+1)(k-3)$

89. $\displaystyle\sum_{i=1}^{4} 2^i$ **90.** $\displaystyle\sum_{j=0}^{4} (-2)^j$

In Exercises 91–94, use a graphing utility to find the sum.

91. $\displaystyle\sum_{j=1}^{6} (24 - 3j)$ **92.** $\displaystyle\sum_{j=1}^{10} \frac{3}{j+1}$

93. $\displaystyle\sum_{k=0}^{4} \frac{(-1)^k}{k+1}$ **94.** $\displaystyle\sum_{k=0}^{4} \frac{(-1)^k}{k!}$

In Exercises 95–104, use sigma notation to write the sum. Then use a graphing utility to find the sum.

95. $\dfrac{1}{3(1)} + \dfrac{1}{3(2)} + \dfrac{1}{3(3)} + \cdots + \dfrac{1}{3(9)}$

96. $\dfrac{5}{1+1}+\dfrac{5}{1+2}+\dfrac{5}{1+3}+\cdots+\dfrac{5}{1+15}$

97. $\left[2\left(\frac{1}{8}\right)+3\right]+\left[2\left(\frac{2}{8}\right)+3\right]+\cdots+\left[2\left(\frac{8}{8}\right)+3\right]$

98. $\left[1-\left(\frac{1}{6}\right)^2\right]+\left[1-\left(\frac{2}{6}\right)^2\right]+\cdots+\left[1-\left(\frac{6}{6}\right)^2\right]$

99. $3-9+27-81+243-729$

100. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots-\frac{1}{128}$

101. $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\cdots-\dfrac{1}{20^2}$

102. $\dfrac{1}{1\cdot 3}+\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 5}+\cdots+\dfrac{1}{10\cdot 12}$

103. $\frac{1}{4}+\frac{3}{8}+\frac{7}{16}+\frac{15}{32}+\frac{31}{64}$

104. $\frac{1}{2}+\frac{2}{4}+\frac{6}{8}+\frac{24}{16}+\frac{120}{32}+\frac{720}{64}$

In Exercises 105–108, find the indicated partial sum of the series.

105. $\displaystyle\sum_{i=1}^{\infty} 5\left(\tfrac{1}{2}\right)^i$

Fourth partial sum

106. $\displaystyle\sum_{i=1}^{\infty} 2\left(\tfrac{1}{3}\right)^i$

Fifth partial sum

107. $\displaystyle\sum_{n=1}^{\infty} 4\left(-\tfrac{1}{2}\right)^n$

Third partial sum

108. $\displaystyle\sum_{n=1}^{\infty} 8\left(-\tfrac{1}{4}\right)^n$

Fourth partial sum

In Exercises 109–112, find (a) the fourth partial sum and (b) the sum of the infinite series.

109. $\displaystyle\sum_{i=1}^{\infty} 6\left(\tfrac{1}{10}\right)^i$

110. $\displaystyle\sum_{k=1}^{\infty} 4\left(\tfrac{1}{10}\right)^k$

111. $\displaystyle\sum_{k=1}^{\infty} \left(\tfrac{1}{10}\right)^k$

112. $\displaystyle\sum_{i=1}^{\infty} 2\left(\tfrac{1}{10}\right)^i$

113. ***Compound Interest*** A deposit of \$5000 is made in an account that earns 3% interest compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 5000\left(1+\frac{0.03}{4}\right)^n, \quad n = 1, 2, 3, \ldots .$$

(a) Compute the first eight terms of this sequence.

(b) Find the balance in this account after 10 years by computing the 40th term of the sequence.

114. ***Compound Interest*** A deposit of \$100 is made *each month* in an account that earns 12% interest compounded monthly. The balance in the account after n months is given by

$$A_n = 100(101)[(1.01)^n - 1], \quad n = 1, 2, 3, \ldots .$$

(a) Compute the first six terms of this sequence.

(b) Find the balance in this account after 5 years by computing the 60th term of the sequence.

(c) Find the balance in this account after 20 years by computing the 240th term of the sequence.

115. ***Fish*** A landlocked lake has been selected to be stocked in the year 2008 with 5500 trout, and to be restocked each year thereafter with 500 trout. Each year the fish population declines 25% due to harvesting and other natural causes.

(a) Write a recursive sequence that gives the population p_n of trout in the lake in terms of the year n since stocking began.

(b) Use the recursion formula from part (a) to find the numbers of trout in the lake in the years 2009, 2010, and 2011.

(c) Use a graphing utility to find the number of trout in the lake as time passes infinitely. Explain your result.

116. ***Tree Farm*** A tree farm in the year 2010 has 10,000 Douglas fir trees on its property. Each year thereafter 10% of the fir trees are harvested and 750 new fir saplings are then planted in their place.

(a) Write a recursive sequence that gives the current number t_n of fir trees on the farm in the year n, with $n = 0$ corresponding to 2010.

(b) Use the recursion formula from part (a) to find the numbers of fir trees for $n = 1, 2, 3,$ and 4. Interpret these values in the context of the situation.

(c) Use a graphing utility to find the number of fir trees as time passes infinitely. Explain your result.

117. ***Investment*** You decide to place \$50 at the beginning of each month into a Roth IRA for your education. The account earns 6% compounded monthly.

(a) Find a recursive sequence that yields the total a_n in the account, where n is the number of months since you began depositing the \$50.

(b) Use the recursion formula from part (a) to find the amount in the IRA after 12 deposits.

(c) Use a graphing utility to find the amount in the IRA after 50 deposits.

Use the *table* feature of a graphing utility to verify your answers in parts (b) and (c).

118. ***Mortgage Payments*** You borrow \$150,000 at 9% interest compounded monthly for 30 years to buy a new home. The monthly mortgage payment has been determined to be \$1206.94.

(a) Find a recursive sequence that gives the balance b_n of the mortgage remaining after each monthly payment n has been made.

(b) Use the *table* feature of a graphing utility to find the balance remaining for every five years where $0 \le n \le 360$.

(c) What is the total amount paid for a \$150,000 loan under these conditions? Explain your answer.

(d) How much interest will be paid over the life of the loan?

119. ***Average Wages*** The average hourly wage rates r_n (in dollars) for instructional teacher aides in the United States from 1998 to 2005 are shown in the table. (Source: Educational Research Service)

Year	Average wage rate, r_n (in dollars)
1998	9.46
1999	9.80
2000	10.00
2001	10.41
2002	10.68
2003	10.93
2004	11.22
2005	11.35

A sequence that models the data is

$$r_n = -0.0092n^2 + 0.491n + 6.11$$

where n is the year, with $n = 8$ corresponding to 1998.

(a) Use a graphing utility to plot the data and graph the model in the same viewing window.

(b) Use the model to find the average hourly wage rates for instructional teacher aides in 2010 and 2015.

(c) Are your results in part (b) reasonable? Explain.

(d) Use the model to find when the average hourly wage rate will reach \$12.

120. ***Education*** The preprimary enrollments s_n (in thousands) in the United States from 1998 to 2003 are shown in the table. (Source: U.S. Census Bureau)

Year	Number of students, s_n (in thousands)
1998	7788
1999	7844
2000	7592
2001	7441
2002	7504
2003	7921

A sequence that models the data is

$$s_n = 33.787n^3 - 1009.56n^2 + 9840.6n - 23{,}613$$

where n is the year, with $n = 8$ corresponding to 1998.

(a) Use a graphing utility to plot the data and graph the model in the same viewing window.

(b) Use the model to find the numbers of students enrolled in a preprimary school in 2005, 2010, and 2015.

(c) Are your results in part (c) reasonable? Explain.

121. ***Revenue*** The revenues R_n (in millions of dollars) for California Pizza Kitchen, Inc. from 1999 to 2006 are shown in the table. (Source: California Pizza Kitchen, Inc.)

Year	Revenue, R_n (in millions of dollars)
1999	179.2
2000	210.8
2001	249.3
2002	306.3
2003	359.9
2004	422.5
2005	480.0
2006	560.0

(a) Use a graphing utility to plot the data. Let n represent the year, with $n = 9$ corresponding to 1999.

(b) Use the *regression* feature of a graphing utility to find a linear sequence and a quadratic sequence that model the data. Identify the coefficient of determination for each model.

(c) Separately graph each model in the same viewing window as the data.

(d) Decide which of the models is a better fit for the data. Explain.

(e) Use the model you chose in part (d) to predict the revenues for the years 2010 and 2015.

(f) Use your model from part (d) to find when the revenues will reach one billion dollars.

122. ***Sales*** The sales S_n (in billions of dollars) for Anheuser-Busch Companies, Inc. from 1995 to 2006 are shown in the table. (Source: Anheuser-Busch Companies, Inc.)

Year	Sales, S_n (in billions of dollars)
1995	10.3
1996	10.9
1997	11.1
1998	11.2
1999	11.7
2000	12.3
2001	12.9
2002	13.6
2003	14.1
2004	14.9
2005	15.1
2006	15.5

(a) Use a graphing utility to plot the data. Let n represent the year, with $n = 5$ corresponding to 1995.

(b) Use the *regression* feature of a graphing utility to find a linear sequence and a quadratic sequence that model the data. Identify the coefficient of determination for each model.

(c) Separately graph each model in the same viewing window as the data.

(d) Decide which of the models is a better fit for the data. Explain.

(e) Use the model you chose in part (d) to predict the sales for Anheuser-Busch for the years 2010 and 2015.

(f) Use your model from part (d) to find when the sales will reach 20 billion dollars.

Synthesis

True or False? **In Exercises 123 and 124, determine whether the statement is true or false. Justify your answer.**

123. $\sum_{i=1}^{4}(i^2 + 2i) = \sum_{i=1}^{4} i^2 + 2\sum_{i=1}^{4} i$

124. $\sum_{j=1}^{4} 2^j = \sum_{j=3}^{6} 2^{j-2}$

Fibonacci Sequence **In Exercises 125 and 126, use the Fibonacci sequence. (See Example 4.)**

125. Write the first 12 terms of the Fibonacci sequence a_n and the first 10 terms of the sequence given by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n > 0.$$

126. Using the definition of b_n given in Exercise 125, show that b_n can be defined recursively by

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

Exploration **In Exercises 127–130, let**

$$a_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

be a sequence with nth term a_n.

127. Use the *table* feature of a graphing utility to find the first five terms of the sequence.

128. Do you recognize the terms of the sequence in Exercise 127? What sequence is it?

129. Find an expression for a_{n+1} and a_{n+2} in terms of n.

130. Use the result from Exercise 129 to show that $a_{n+2} = a_{n+1} + a_n$. Is this result the same as your answer to Exercise 127? Explain.

FOR FURTHER INFORMATION: Use the Internet or your library to read more about the Fibonacci sequence in the publication called the *Fibonacci Quarterly*, a journal dedicated to this famous result in mathematics.

In Exercises 131–140, write the first five terms of the sequence.

131. $a_n = \dfrac{x^n}{n!}$

132. $a_n = \dfrac{x^2}{n^2}$

133. $a_n = \dfrac{(-1)^n x^{2n+1}}{2n + 1}$

134. $a_n = \dfrac{(-1)^n x^{n+1}}{n + 1}$

135. $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$

136. $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n + 1)!}$

137. $a_n = \dfrac{(-1)^n x^n}{n!}$

138. $a_n = \dfrac{(-1)^n x^{n+1}}{(n + 1)!}$

139. $a_n = \dfrac{(-1)^{n+1}(x + 1)^n}{n!}$

140. $a_n = \dfrac{(-1)^n(x - 1)^n}{(n + 1)!}$

In Exercises 141–146, write the first five terms of the sequence. Then find an expression for the nth partial sum.

141. $a_n = \dfrac{1}{2n} - \dfrac{1}{2n + 2}$

142. $a_n = \dfrac{1}{n} - \dfrac{1}{n + 1}$

143. $a_n = \dfrac{1}{n + 1} - \dfrac{1}{n + 2}$

144. $a_n = \dfrac{1}{n} - \dfrac{1}{n + 2}$

145. $a_n = \ln n$

146. $a_n = 1 - \ln(n + 1)$

Skills Review

In Exercises 147–150, find, if possible, (a) $A - B$, (b) $2B - 3A$, (c) AB, and (d) BA.

147. $A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 6 & -3 \end{bmatrix}$

148. $A = \begin{bmatrix} 10 & 7 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -12 \\ 8 & 11 \end{bmatrix}$

149. $A = \begin{bmatrix} -2 & -3 & 6 \\ 4 & 5 & 7 \\ 1 & 7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 6 \\ 0 & 3 & 1 \end{bmatrix}$

150. $A = \begin{bmatrix} -1 & 4 & 0 \\ 5 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

8.2 Arithmetic Sequences and Partial Sums

Arithmetic Sequences

A sequence whose consecutive terms have a common difference is called an **arithmetic sequence.**

Definition of Arithmetic Sequence

A sequence is **arithmetic** if the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is arithmetic if there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = d.$$

The number d is the **common difference** of the arithmetic sequence.

What you should learn

- Recognize, write, and find the nth terms of arithmetic sequences.
- Find nth partial sums of arithmetic sequences.
- Use arithmetic sequences to model and solve real-life problems.

Why you should learn it

Arithmetic sequences can reduce the amount of time it takes to find the sum of a sequence of numbers with a common difference. In Exercise 81 on page 599, you will use an arithmetic sequence to find the number of bricks needed to lay a brick patio.

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Example 1 Examples of Arithmetic Sequences

a. The sequence whose nth term is $4n + 3$ is arithmetic. For this sequence, the common difference between consecutive terms is 4.

$$\underbrace{7, 11}_{11 - 7 = 4}, 15, 19, \ldots, 4n + 3, \ldots \qquad \text{Begin with } n = 1.$$

b. The sequence whose nth term is $7 - 5n$ is arithmetic. For this sequence, the common difference between consecutive terms is -5.

$$\underbrace{2, -3}_{-3 - 2 = -5}, -8, -13, \ldots, 7 - 5n, \ldots \qquad \text{Begin with } n = 1.$$

c. The sequence whose nth term is $\frac{1}{4}(n + 3)$ is arithmetic. For this sequence, the common difference between consecutive terms is $\frac{1}{4}$.

$$\underbrace{1, \frac{5}{4}}_{\frac{5}{4} - 1 = \frac{1}{4}}, \frac{3}{2}, \frac{7}{4}, \ldots, \frac{n + 3}{4}, \ldots \qquad \text{Begin with } n = 1.$$

CHECKPOINT Now try Exercise 9.

The sequence 1, 4, 9, 16, . . . , whose nth term is n^2, is *not* arithmetic. The difference between the first two terms is

$$a_2 - a_1 = 4 - 1 = 3$$

but the difference between the second and third terms is

$$a_3 - a_2 = 9 - 4 = 5.$$

In Example 1, notice that each of the arithmetic sequences has an nth term that is of the form $dn + c$, where the common difference of the sequence is d.

The nth Term of an Arithmetic Sequence

The nth term of an arithmetic sequence has the form

$$a_n = dn + c$$

where d is the common difference between consecutive terms of the sequence and $c = a_1 - d$.

An arithmetic sequence $a_n = dn + c$ can be thought of as "counting by d's" after a shift of c units from d. For instance, the sequence

$2, 6, 10, 14, 18, . . .$

has a common difference of 4, so you are counting by 4's after a shift of two units below 4 (beginning with $a_1 = 2$). So, the nth term is $4n - 2$. Similarly, the nth term of the sequence

$6, 11, 16, 21, . . .$

is $5n + 1$ because you are counting by 5's after a shift of one unit above 5 (beginning with $a_1 = 6$).

Example 2 Finding the nth Term of an Arithmetic Sequence

Find a formula for the nth term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

Solution

Because the sequence is arithmetic, you know that the formula for the nth term is of the form $a_n = dn + c$. Moreover, because the common difference is $d = 3$, the formula must have the form $a_n = 3n + c$. Because $a_1 = 2$, it follows that

$$c = a_1 - d = 2 - 3 = -1.$$

So, the formula for the nth term is $a_n = 3n - 1$. The sequence therefore has the following form.

$2, 5, 8, 11, 14, . . . , 3n - 1, . . .$

A graph of the first 15 terms of the sequence is shown in Figure 8.8. Notice that the points lie on a line. This makes sense because a_n is a linear function of n. In other words, the terms "arithmetic" and "linear" are closely connected.

CHECKPOINT Now try Exercise 17.

Another way to find a formula for the nth term of the sequence in Example 2 is to begin by writing the terms of the sequence.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	. . .
2	2 + 3	5 + 3	8 + 3	11 + 3	14 + 3	17 + 3	. . .
2	5	8	11	14	17	20	. . .

So, you can reason that the nth term is of the form

$$a_n = dn + c = 3n - 1.$$

Exploration

Consider the following sequences.

$1, 4, 7, 10, 13, . . . ,$

$3n - 2, . . . ;$

$-5, 1, 7, 13, 19, . . . ,$

$6n - 11, . . . ;$

$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, . . . , \frac{7}{2} - n, . . .$

What relationship do you observe between successive terms of these sequences?

TECHNOLOGY TIP

You can use a graphing utility to generate the arithmetic sequence in Example 2 by using the following steps.

2 (ENTER)

3 (+) (ANS)

Now press the enter key repeatedly to generate the terms of the sequence.

Most graphing utilities have a built-in function that will display the terms of an arithmetic sequence. For instructions on how to use the *sequence* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center.*

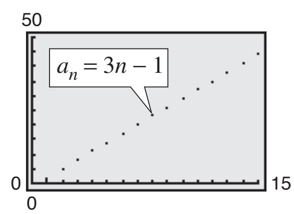


Figure 8.8

Example 3 Writing the Terms of an Arithmetic Sequence

The fourth term of an arithmetic sequence is 20, and the 13th term is 65. Write the first several terms of this sequence.

Solution

The fourth and 13th terms of the sequence are related by

$$a_{13} = a_4 + 9d.$$

Using $a_4 = 20$ and $a_{13} = 65$, you have $65 = 20 + 9d$. So, you can conclude that $d = 5$, which implies that the sequence is as follows.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	. . .
5,	10,	15,	20,	25,	30,	35,	40,	45,	50,	55,	60,	65,	. . .

CHECKPOINT Now try Exercise 31.

STUDY TIP

In Example 3, the relationship between the fourth and 13th terms can be found by subtracting the equation for the fourth term, $a_4 = 4d + c$, from the equation for the 13th term, $a_{13} = 13d + c$. The result, $a_{13} - a_4 = 9d$, can be rewritten as $a_{13} = a_4 + 9d$.

If you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$th term by using the *recursion formula*

$$a_{n+1} = a_n + d. \quad \text{Recursion formula}$$

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the preceding term. For instance, if you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

Example 4 Using a Recursion Formula

Find the seventh term of the arithmetic sequence whose first two terms are 2 and 9.

Solution

For this sequence, the common difference is $d = 9 - 2 = 7$. Next find a formula for the nth term. Because the first term is 2, it follows that

$$c = a_1 - d = 2 - 7 = -5.$$

Therefore, a formula for the nth term is

$$a_n = dn + c = 7n - 5.$$

which implies that the seventh term is

$$a_7 = 7(7) - 5 = 44.$$

CHECKPOINT Now try Exercise 39.

STUDY TIP

Another way to find the seventh term in Example 4 is to determine the common difference, $d = 7$, and then simply write out the first seven terms (by repeatedly adding 7).

2, 9, 16, 23, 30, 37, 44

As you can see, the seventh term is 44.

If you substitute $a_1 - d$ for c in the formula $a_n = dn + c$, the nth term of an arithmetic sequence has the alternative recursion formula

$$a_n = a_1 + (n - 1)d. \quad \text{Alternative recursion formula}$$

Use this formula to solve Example 4. You should obtain the same answer.

The Sum of a Finite Arithmetic Sequence

There is a simple formula for the *sum* of a finite arithmetic sequence.

The Sum of a Finite Arithmetic Sequence (See the proof on page 657.)

The sum of a finite arithmetic sequence with n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Be sure you see that this formula works only for *arithmetic* sequences. Using this formula reduces the amount of time it takes to find the sum of an arithmetic sequence, as you will see in the following example.

Example 5 Finding the Sum of a Finite Arithmetic Sequence

Find each sum.

a. $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$

b. Sum of the integers from 1 to 100

Solution

a. To begin, notice that the sequence is arithmetic (with a common difference of 2). Moreover, the sequence has 10 terms. So, the sum of the sequence is

$$S_n = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$$

$$= \frac{n}{2}(a_1 + a_n) \qquad \text{Formula for sum of an arithmetic sequence}$$

$$= \frac{10}{2}(1 + 19) \qquad \text{Substitute 10 for } n\text{, 1 for } a_1\text{, and 19 for } a_n.$$

$$= 5(20) = 100. \qquad \text{Simplify.}$$

b. The integers from 1 to 100 form an arithmetic sequence that has 100 terms. So, you can use the formula for the sum of an arithmetic sequence, as follows.

$$S_n = 1 + 2 + 3 + 4 + 5 + 6 + \cdot \cdot \cdot + 99 + 100$$

$$= \frac{n}{2}(a_1 + a_n) \qquad \text{Formula for sum of an arithmetic sequence}$$

$$= \frac{100}{2}(1 + 100) \qquad \text{Substitute 100 for } n\text{, 1 for } a_1\text{, and 100 for } a_n.$$

$$= 50(101) = 5050 \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 53.

The sum of the first n terms of an infinite sequence is called the ***n*th partial sum.** The nth partial sum of an arithmetic sequence can be found by using the formula for the sum of a finite arithmetic sequence.

Example 6 Finding a Partial Sum of an Arithmetic Sequence

Find the 150th partial sum of the arithmetic sequence 5, 16, 27, 38, 49, ...

Solution

For this arithmetic sequence, you have $a_1 = 5$ and $d = 16 - 5 = 11$. So,

$$c = a_1 - d = 5 - 11 = -6$$

and the nth term is $a_n = 11n - 6$. Therefore, $a_{150} = 11(150) - 6 = 1644$, and the sum of the first 150 terms is

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{nth partial sum formula}$$

$$= \frac{150}{2}(5 + 1644) \qquad \text{Substitute 150 for n, 5 for a_1, and 1644 for a_n.}$$

$$= 75(1649) = 123{,}675. \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 61.

Applications

Example 7 Seating Capacity

An auditorium has 20 rows of seats. There are 20 seats in the first row, 21 seats in the second row, 22 seats in the third row, and so on (see Figure 8.9). How many seats are there in all 20 rows?

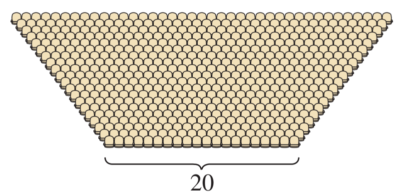


Figure 8.9

Solution

The numbers of seats in the 20 rows form an arithmetic sequence for which the common difference is $d = 1$. Because

$$c = a_1 - d = 20 - 1 = 19$$

you can determine that the formula for the nth term of the sequence is $a_n = n + 19$. So, the 20th term of the sequence is $a_{20} = 20 + 19 = 39$, and the total number of seats is

$$S_n = 20 + 21 + 22 + \cdots + 39$$

$$= \frac{20}{2}(20 + 39) \qquad \text{Substitute 20 for n, 20 for a_1, and 39 for a_n.}$$

$$= 10(59) = 590. \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 81.

Example 8 Total Sales

A small business sells \$10,000 worth of sports memorabilia during its first year. The owner of the business has set a goal of increasing annual sales by \$7500 each year for 19 years. Assuming that this goal is met, find the total sales during the first 20 years this business is in operation.

Algebraic Solution

The annual sales form an arithmetic sequence in which $a_1 = 10{,}000$ and $d = 7500$. So,

$$\begin{aligned} c &= a_1 - d \\ &= 10{,}000 - 7500 \\ &= 2500 \end{aligned}$$

and the nth term of the sequence is

$$a_n = 7500n + 2500.$$

This implies that the 20th term of the sequence is

$$\begin{aligned} a_{20} &= 7500(20) + 2500 \\ &= 152{,}500. \end{aligned}$$

The sum of the first 20 terms of the sequence is

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) && n\text{th partial sum formula} \\ &= \frac{20}{2}(10{,}000 + 152{,}500) && \text{Substitute 20 for } n, 10{,}000 \text{ for } a_1, \text{ and } 152{,}500 \text{ for } a_n. \\ &= 10(162{,}500) && \text{Simplify.} \\ &= 1{,}625{,}000. && \text{Simplify.} \end{aligned}$$

So, the total sales for the first 20 years are \$1,625,000.

CHECKPOINT Now try Exercise 83.

Numerical Solution

The annual sales form an arithmetic sequence in which $a_1 = 10{,}000$ and $d = 7500$. So,

$$\begin{aligned} c &= a_1 - d \\ &= 10{,}000 - 7500 \\ &= 2500. \end{aligned}$$

The nth term of the sequence is given by

$$u_n = 7500n + 2500.$$

You can use the *list editor* of a graphing utility to create a table that shows the sales for each of the 20 years. First, enter the numbers 1 through 20 in L_1. Then enter $7500{*}L_1 + 2500$ for L_2. You should obtain a table like the one shown in Figure 8.10. Finally, use the *sum* feature of the graphing utility to find the sum of the data in L_2, as shown in Figure 8.11. So, the total sales for the first 20 years are \$1,625,000.

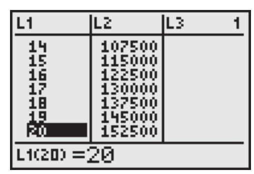


Figure 8.10

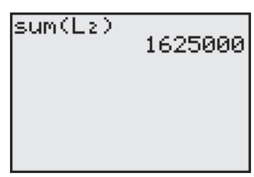


Figure 8.11

If you go on to take a course in calculus, you will study sequences and series in detail. You will learn that sequences and series play a major role in the study of calculus.

TECHNOLOGY SUPPORT

For instructions on how to use the *list editor* and *sum* features, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center.*

8.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. A sequence is called an _______ sequence if the differences between consecutive terms are the same. This difference is called the _______ difference.
2. The nth term of an arithmetic sequence has the form _______ .
3. The formula $S_n = \frac{n}{2}(a_1 + a_n)$ can be used to find the sum of the first n terms of an arithmetic sequence, called the _______ .

In Exercises 1–8, determine whether or not the sequence is arithmetic. If it is, find the common difference.

1. $10, 8, 6, 4, 2, \ldots$
2. $4, 9, 14, 19, 24, \ldots$
3. $3, \frac{5}{2}, 2, \frac{3}{2}, 1, \ldots$
4. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \ldots$
5. $-24, -16, -8, 0, 8, \ldots$
6. $\ln 1, \ln 2, \ln 3, \ln 4, \ln 5, \ldots$
7. $3.7, 4.3, 4.9, 5.5, 6.1, \ldots$
8. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

In Exercises 9–16, write the first five terms of the sequence. Determine whether or not the sequence is arithmetic. If it is, find the common difference. (Assume n begins with 1.)

9. $a_n = 8 + 13n$
10. $a_n = 2^n + n$
11. $a_n = \dfrac{1}{n+1}$
12. $a_n = 1 + (n-1)4$
13. $a_n = 150 - 7n$
14. $a_n = 2^{n-1}$
15. $a_n = 3 + 2(-1)^n$
16. $a_n = 3 - 4(n + 6)$

In Exercises 17–26, find a formula for a_n for the arithmetic sequence.

17. $a_1 = 1, d = 3$
18. $a_1 = 15, d = 4$
19. $a_1 = 100, d = -8$
20. $a_1 = 0, d = -\frac{2}{3}$
21. $4, \frac{3}{2}, -1, -\frac{7}{2}, \ldots$
22. $10, 5, 0, -5, -10, \ldots$
23. $a_1 = 5, a_4 = 15$
24. $a_1 = -4, a_5 = 16$
25. $a_3 = 94, a_6 = 85$
26. $a_5 = 190, a_{10} = 115$

In Exercises 27–34, write the first five terms of the arithmetic sequence. Use the *table* feature of a graphing utility to verify your results.

27. $a_1 = 5, d = 6$
28. $a_1 = 5, d = -\frac{3}{4}$
29. $a_1 = -10, d = -12$
30. $a_4 = 16, a_{10} = 46$
31. $a_8 = 26, a_{12} = 42$
32. $a_6 = -38, a_{11} = -73$
33. $a_3 = 19, a_{15} = -1.7$
34. $a_5 = 16, a_{14} = 38.5$

In Exercises 35–38, write the first five terms of the arithmetic sequence. Find the common difference and write the nth term of the sequence as a function of n.

35. $a_1 = 15, \quad a_{k+1} = a_k + 4$
36. $a_1 = 200, \quad a_{k+1} = a_k - 10$
37. $a_1 = \frac{3}{5}, \quad a_{k+1} = -\frac{1}{10} + a_k$
38. $a_1 = 1.5, \quad a_{k+1} = a_k - 2.5$

In Exercises 39–42, the first two terms of the arithmetic sequence are given. Find the missing term. Use the *table* feature of a graphing utility to verify your results.

39. $a_1 = 5, \quad a_2 = 11, \quad a_{10} =$ ▭
40. $a_1 = 3, \quad a_2 = 13, \quad a_9 =$ ▭
41. $a_1 = 4.2, \quad a_2 = 6.6, \quad a_7 =$ ▭
42. $a_1 = -0.7, \quad a_2 = -13.8, \quad a_8 =$ ▭

In Exercises 43–46, use a graphing utility to graph the first 10 terms of the sequence. (Assume n begins with 1.)

43. $a_n = 15 - \frac{3}{2}n$
44. $a_n = -5 + 2n$
45. $a_n = 0.5n + 4$
46. $a_n = -0.9n + 2$

In Exercises 47–52, use the *table* feature of a graphing utility to find the first 10 terms of the sequence. (Assume n begins with 1.)

47. $a_n = 4n - 5$
48. $a_n = 17 + 3n$
49. $a_n = 20 - \frac{3}{4}n$
50. $a_n = \frac{4}{5}n + 12$
51. $a_n = 1.5 + 0.05n$
52. $a_n = 8 - 12.5n$

In Exercises 53–60, find the sum of the finite arithmetic sequence.

53. $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$
54. $1 + 4 + 7 + 10 + 13 + 16 + 19$
55. $-1 + (-3) + (-5) + (-7) + (-9)$

56. $-5 + (-3) + (-1) + 1 + 3 + 5$

57. Sum of the first 50 positive even integers

58. Sum of the first 100 positive odd integers

59. Sum of the integers from -100 to 30

60. Sum of the integers from -10 to 50

In Exercises 61–66, find the indicated *n*th partial sum of the arithmetic sequence.

61. 8, 20, 32, 44, . . . , $n = 10$

62. $-6, -2, 2, 6, \ldots, \quad n = 50$

63. 0.5, 1.3, 2.1, 2.9, . . . , $n = 10$

64. 4.2, 3.7, 3.2, 2.7, . . . , $n = 12$

65. $a_1 = 100, \quad a_{25} = 220, \quad n = 25$

66. $a_1 = 15, \quad a_{100} = 307, \quad n = 100$

In Exercises 67–74, find the partial sum without using a graphing utility.

67. $\sum_{n=1}^{50} n$

68. $\sum_{n=1}^{100} 2n$

69. $\sum_{n=1}^{100} 5n$

70. $\sum_{n=51}^{100} 7n$

71. $\sum_{n=11}^{30} n - \sum_{n=1}^{10} n$

72. $\sum_{n=51}^{100} n - \sum_{n=1}^{50} n$

73. $\sum_{n=1}^{500} (n + 8)$

74. $\sum_{n=1}^{250} (1000 - n)$

In Exercises 75–80, use a graphing utility to find the partial sum.

75. $\sum_{n=1}^{20} (2n + 1)$

76. $\sum_{n=0}^{50} (50 - 2n)$

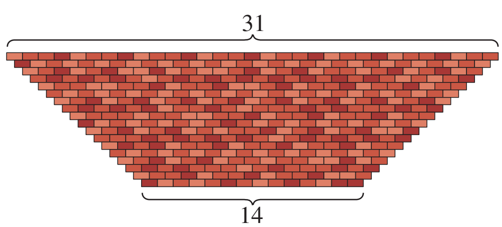
77. $\sum_{n=1}^{100} \frac{n + 1}{2}$

78. $\sum_{n=0}^{100} \frac{4 - n}{4}$

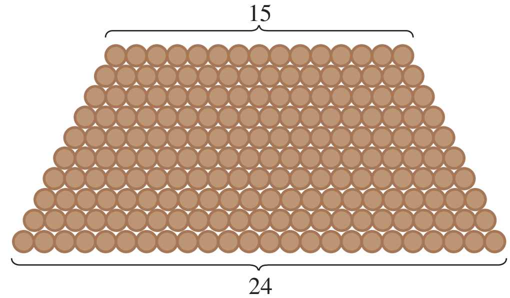
79. $\sum_{i=1}^{60} \left(250 - \frac{2}{5}i\right)$

80. $\sum_{j=1}^{200} (10.5 + 0.025j)$

81. ***Brick Pattern*** A brick patio has the approximate shape of a trapezoid, as shown in the figure. The patio has 18 rows of bricks. The first row has 14 bricks and the 18th row has 31 bricks. How many bricks are in the patio?



82. ***Number of Logs*** Logs are stacked in a pile, as shown in the figure. The top row has 15 logs and the bottom row has 24 logs. How many logs are in the pile?



83. ***Sales*** A small hardware store makes a profit of $20,000 during its first year. The store owner sets a goal of increasing profits by $5000 each year for 4 years. Assuming that this goal is met, find the total profit during the first 5 years of business.

84. ***Falling Object*** An object with negligible air resistance is dropped from an airplane. During the first second of fall, the object falls 4.9 meters; during the second second, it falls 14.7 meters; during the third second, it falls 24.5 meters; and during the fourth second, it falls 34.3 meters. If this arithmetic pattern continues, how many meters will the object fall in 10 seconds?

85. ***Sales*** The table shows the sales a_n (in billions of dollars) for Coca-Cola Enterprises, Inc. from 1997 to 2004. (Source: Coca-Cola Enterprises, Inc.)

Year	Sales, a_n (in billions of dollars)
1997	11.3
1998	13.4
1999	14.4
2000	14.8
2001	15.7
2002	16.9
2003	17.3
2004	18.2

(a) Use the *regression* feature of a graphing utility to find an arithmetic sequence for the data. Let n represent the year, with $n = 7$ corresponding to 1997.

(b) Use the sequence from part (a) to approximate the annual sales for Coca-Cola Enterprises, Inc. for the years 1997 to 2004. How well does the model fit the data?

(c) Use the sequence to find the total annual sales for Coca Cola for the years from 1997 to 2004.

(d) Use the sequence to predict the total annual sales for the years 2005 to 2012. Is your total reasonable? Explain.

86. ***Education*** The table shows the numbers a_n (in thousands) of master's degrees conferred in the United States from 1995 to 2003. (Source: U.S. National Center for Education Statistics)

Year	Master's degrees conferred, a_n (in thousands)
1995	398
1996	406
1997	419
1998	430
1999	440
2000	457
2001	468
2002	482
2003	512

(a) Use the *regression* feature of a graphing utility to find an arithmetic sequence for the data. Let n represent the year, with $n = 5$ corresponding to 1995.

(b) Use the sequence from part (a) to approximate the numbers of master's degrees conferred for the years 1995 to 2003. How well does the model fit the data?

(c) Use the sequence to find the total number of master's degrees conferred over the period from 1995 to 2003.

(d) Use the sequence to predict the total number of master's degrees conferred over the period from 2004 to 2014. Is your total reasonable? Explain.

Synthesis

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.**

87. Given an arithmetic sequence for which only the first and second terms are known, it is possible to find the nth term.

88. If the only known information about a finite arithmetic sequence is its first term and its last term, then it is possible to find the sum of the sequence.

In Exercises 89 and 90, find the first 10 terms of the sequence.

89. $a_1 = x, d = 2x$ **90.** $a_1 = -y, d = 5y$

91. ***Think About It*** The sum of the first 20 terms of an arithmetic sequence with a common difference of 3 is 650. Find the first term.

92. ***Think About It*** The sum of the first n terms of an arithmetic sequence with first term a_1 and common difference d is S_n. Determine the sum if each term is increased by 5. Explain.

93. ***Think About It*** Decide whether it is possible to fill in the blanks in each of the sequences such that the resulting sequence is arithmetic. If so, find a recursion formula for the sequence. Write a short paragraph explaining how you made your decisions.

(a) -7, $\square$, $\square$, $\square$, $\square$, $\square$, 11

(b) 17, $\square$, $\square$, $\square$, $\square$, $\square$, $\square$, 59

(c) 2, 6, $\square$, $\square$, 162

(d) 4, 7.5, $\square$, $\square$, $\square$, $\square$, $\square$, 28.5

(e) 8, 12, $\square$, $\square$, $\square$, 60.75

94. ***Gauss*** Carl Friedrich Gauss, a famous nineteenth century mathematician, was a child prodigy. It was said that when Gauss was 10 he was asked by his teacher to add the numbers from 1 to 100. Almost immediately, Gauss found the answer by mentally finding the summation. Write an explanation of how he arrived at his conclusion, and then find the formula for the sum of the first n natural numbers.

In Exercises 95–98, find the sum using the method from Exercise 94.

95. The first 200 natural numbers

96. The first 100 even natural numbers from 2 to 200, inclusive

97. The first 51 odd natural numbers from 1 to 101, inclusive

98. The first 100 multiples of 4 from 4 to 400, inclusive

Skills Review

In Exercises 99 and 100, use Gauss-Jordan elimination to solve the system of equations.

99. $\begin{cases} 2x - y + 7z = -10 \\ 3x + 2y - 4z = 17 \\ 6x - 5y + z = -20 \end{cases}$ **100.** $\begin{cases} -x + 4y + 10z = 4 \\ 5x - 3y + z = 31 \\ 8x + 2y - 3z = -5 \end{cases}$

In Exercises 101 and 102, use a determinant to find the area of the triangle with the given vertices.

101. $(0, 0), (4, -3), (2, 6)$ **102.** $(-1, 2), (5, 1), (3, 8)$

103. ***Make a Decision*** To work an extended application analyzing the amount of municipal waste recovered in the United States from 1983 to 2003, visit this textbook's *Online Study Center.* *(Data Source: U.S. Census Bureau)*

8.3 Geometric Sequences and Series

Geometric Sequences

In Section 8.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence.** Consecutive terms of a geometric sequence have a common *ratio*.

Definition of Geometric Sequence

A sequence is **geometric** if the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is geometric if there is a number r such that

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = r, \qquad r \neq 0.$$

The number r is the **common ratio** of the sequence.

What you should learn

- Recognize, write, and find the *n*th terms of geometric sequences.
- Find *n*th partial sums of geometric sequences.
- Find sums of infinite geometric series.
- Use geometric sequences to model and solve real-life problems.

Why you should learn it

Geometric sequences can reduce the amount of time it takes to find the sum of a sequence of numbers with a common ratio. For instance, Exercise 99 on page 610 shows how to use a geometric sequence to estimate the distance a bungee jumper travels after jumping off a bridge.

Brand X Pictures/age fotostock

Example 1 Examples of Geometric Sequences

a. The sequence whose nth term is 2^n is geometric. For this sequence, the common ratio between consecutive terms is 2.

$$\underbrace{2, 4}_{\frac{4}{2} = 2}, 8, 16, \ldots, 2^n, \ldots \qquad \text{Begin with } n = 1.$$

b. The sequence whose nth term is $4(3^n)$ is geometric. For this sequence, the common ratio between consecutive terms is 3.

$$\underbrace{12, 36}_{\frac{36}{12} = 3}, 108, 324, \ldots, 4(3^n), \ldots \qquad \text{Begin with } n = 1.$$

c. The sequence whose nth term is $\left(-\frac{1}{3}\right)^n$ is geometric. For this sequence, the common ratio between consecutive terms is $-\frac{1}{3}$.

$$\underbrace{-\frac{1}{3}, \frac{1}{9}}_{\frac{1/9}{-1/3} = -\frac{1}{3}}, -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots \qquad \text{Begin with } n = 1.$$

CHECKPOINT Now try Exercise 1.

The sequence $1, 4, 9, 16, \ldots$, whose nth term is n^2, is *not* geometric. The ratio of the second term to the first term is

$$\frac{a_2}{a_1} = \frac{4}{1} = 4$$

but the ratio of the third term to the second term is $\dfrac{a_3}{a_2} = \dfrac{9}{4}$.

STUDY TIP

In Example 1, notice that each of the geometric sequences has an nth term of the form ar^n, where r is the common ratio of the sequence.

The nth Term of a Geometric Sequence

The nth term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$\begin{array}{ccccccc} a_1, & a_2, & a_3, & a_4, & a_5, & \ldots, & a_n, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ a_1, & a_1r, & a_1r^2, & a_1r^3, & a_1r^4, & \ldots, & a_1r^{n-1}, & \ldots \end{array}$$

If you know the nth term of a geometric sequence, you can find the $(n + 1)$th term by multiplying by r. That is, $a_{n+1} = a_n r$.

Example 2 Finding the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$.

Solution

Starting with 3, repeatedly multiply by 2 to obtain the following.

$a_1 = 3$	1st term	$a_4 = 3(2^3) = 24$	4th term
$a_2 = 3(2^1) = 6$	2nd term	$a_5 = 3(2^4) = 48$	5th term
$a_3 = 3(2^2) = 12$	3rd term		

CHECKPOINT Now try Exercise 11.

TECHNOLOGY TIP

You can use a graphing utility to generate the geometric sequence in Example 2 by using the following steps.

3 [ENTER]

2 [×] [ANS]

Now press the *enter* key repeatedly to generate the terms of the sequence.

Most graphing utilities have a built-in function that will display the terms of a geometric sequence.

Example 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

Algebraic Solution

$$a_n = a_1 r^{n-1} \qquad \text{Formula for a geometric sequence}$$

$$a_{15} = 20(1.05)^{15-1} \qquad \text{Substitute 20 for } a_1, 1.05 \text{ for } r, \text{ and 15 for } n.$$

$$\approx 39.60 \qquad \text{Use a calculator.}$$

CHECKPOINT Now try Exercise 25.

Numerical Solution

For this sequence, $r = 1.05$ and $a_1 = 20$. So, $a_n = 20(1.05)^{n-1}$. Use the *table* feature of a graphing utility to create a table that shows the values of $u_n = 20(1.05)^{n-1}$ for $n = 1$ through $n = 15$. From Figure 8.12, the number in the 15th row is approximately 39.60, so the 15th term of the geometric sequence is about 39.60.

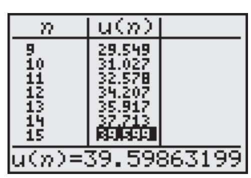


Figure 8.12

Example 4 Finding a Term of a Geometric Sequence

Find a formula for the nth term of the following geometric sequence. What is the ninth term of the sequence?

5, 15, 45, . . .

Solution

The common ratio of this sequence is

$$r = \frac{15}{5} = 3.$$

Because the first term is $a_1 = 5$, the formula must have the form

$$a_n = a_1 r^{n-1} = 5(3)^{n-1}.$$

You can determine the ninth term $(n = 9)$ to be

$$\begin{aligned} a_9 &= 5(3)^{9-1} && \text{Substitute 9 for } n. \\ &= 5(6561) = 32{,}805. && \text{Simplify.} \end{aligned}$$

A graph of the first nine terms of the sequence is shown in Figure 8.13. Notice that the points lie on an exponential curve. This makes sense because a_n is an exponential function of n.

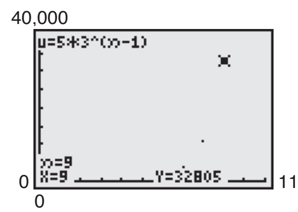


Figure 8.13

CHECKPOINT Now try Exercise 33.

If you know *any* two terms of a geometric sequence, you can use that information to find a formula for the nth term of the sequence.

Example 5 Finding a Term of a Geometric Sequence

The fourth term of a geometric sequence is 125, and the 10th term is 125/64. Find the 14th term. (Assume that the terms of the sequence are positive.)

Solution

The 10th term is related to the fourth term by the equation

$$a_{10} = a_4 r^6. \qquad \text{Multiply 4th term by } r^{10-4}.$$

Because $a_{10} = 125/64$ and $a_4 = 125$, you can solve for r as follows.

$$\frac{125}{64} = 125r^6$$

$$\frac{1}{64} = r^6 \quad \Rightarrow \quad \frac{1}{2} = r$$

You can obtain the 14th term by multiplying the 10th term by r^4.

$$a_{14} = a_{10} r^4 = \frac{125}{64}\left(\frac{1}{2}\right)^4 = \frac{125}{1024}$$

CHECKPOINT Now try Exercise 31.

STUDY TIP

Remember that r is the common ratio of consecutive terms of a geometric sequence. So, in Example 5,

$$\begin{aligned} a_{10} &= a_1 r^9 \\ &= a_1 \cdot r \cdot r \cdot r \cdot r^6 \\ &= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6 \\ &= a_4 r^6. \end{aligned}$$

The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

The Sum of a Finite Geometric Sequence (See the proof on page 657.)

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4, \ldots, a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right).$$

Example 6 Finding the Sum of a Finite Geometric Sequence

Find the sum $\displaystyle\sum_{n=1}^{12} 4(0.3)^n$.

Solution

By writing out a few terms, you have

$$\sum_{n=1}^{12} 4(0.3)^n = 4(0.3)^1 + 4(0.3)^2 + 4(0.3)^3 + \cdots + 4(0.3)^{12}.$$

Now, because $a_1 = 4(0.3)$, $r = 0.3$, and $n = 12$, you can apply the formula for the sum of a finite geometric sequence to obtain

$$\sum_{n=1}^{12} 4(0.3)^n = a_1\left(\frac{1-r^n}{1-r}\right)$$ Formula for sum of a finite geometric sequence

$$= 4(0.3)\left[\frac{1-(0.3)^{12}}{1-0.3}\right]$$ Substitute $4(0.3)$ for a_1, 0.3 for r, and 12 for n.

$$\approx 1.71.$$ Use a calculator.

CHECKPOINT Now try Exercise 45.

When using the formula for the sum of a geometric sequence, be careful to check that the index begins at $i = 1$. If the index begins at $i = 0$, you must adjust the formula for the nth partial sum. For instance, if the index in Example 6 had begun with $n = 0$, the sum would have been

$$\sum_{n=0}^{12} 4(0.3)^n = 4(0.3)^0 + \sum_{n=1}^{12} 4(0.3)^n$$

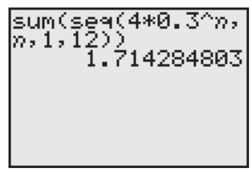
$$= 4 + \sum_{n=1}^{12} 4(0.3)^n$$

$$\approx 4 + 1.71$$

$$= 5.71.$$

TECHNOLOGY TIP

Using the *sum sequence* feature of a graphing utility, you can calculate the sum of the sequence in Example 6 to be about 1.7142848, as shown below.



Calculate the sum beginning at $n = 0$. You should obtain a sum of 5.7142848.

Geometric Series

The sum of the terms of an infinite geometric sequence is called an **infinite geometric series** or simply a **geometric series.**

The formula for the sum of a *finite geometric sequence* can, depending on the value of r, be extended to produce a formula for the sum of an *infinite geometric series*. Specifically, if the common ratio r has the property that $|r| < 1$, it can be shown that r^n becomes arbitrarily close to zero as n increases without bound. Consequently,

$$a_1\left(\frac{1 - r^n}{1 - r}\right) \longrightarrow a_1\left(\frac{1 - 0}{1 - r}\right) \quad \text{as} \quad n \longrightarrow \infty.$$

This result is summarized as follows.

The Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdot \cdot \cdot + a_1 r^{n-1} + \cdot \cdot \cdot$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1 r^i = \frac{a_1}{1 - r}.$$

Note that if $|r| \geq 1$, the series does not have a sum.

Exploration

Notice that the formula for the sum of an infinite geometric series requires that $|r| < 1$. What happens if $r = 1$ or $r = -1$? Give examples of infinite geometric series for which $|r| > 1$ and convince yourself that they do not have finite sums.

Example 7 Finding the Sum of an Infinite Geometric Series

Use a graphing utility to find the first six partial sums of the series. Then find the sum of the series.

$$\sum_{n=1}^{\infty} 4(0.6)^{n-1}$$

Solution

You can use the *cumulative sum* feature to find the first six partial sums of the series, as shown in Figure 8.14. By scrolling to the right, you can determine that the first six partial sums are as follows.

4, 6.4, 7.84, 8.704, 9.2224, 9.53344

Use the formula for the sum of an infinite geometric series to find the sum.

$$\sum_{n=1}^{\infty} 4(0.6)^{n-1} = 4(1) + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdot \cdot \cdot + 4(0.6)^{n-1} + \cdot \cdot \cdot$$

$$= \frac{4}{1 - 0.6} = 10 \qquad \frac{a_1}{1 - r}$$

CHECKPOINT Now try Exercise 65.

TECHNOLOGY SUPPORT

For instructions on how to use the *cumulative sum* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

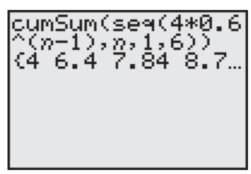


Figure 8.14

Example 8 Finding the Sum of an Infinite Geometric Series

Find the sum $3 + 0.3 + 0.03 + 0.003 + \cdots$.

Solution

$$\begin{aligned} 3 + 0.3 + 0.03 + 0.003 + \cdots &= 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots \\ &= \frac{3}{1 - 0.1} \qquad \frac{a_1}{1 - r} \\ &= \frac{10}{3} \\ &\approx 3.33 \end{aligned}$$

CHECKPOINT Now try Exercise 69.

Exploration

Notice in Example 7 that when using a graphing utility to find the sum of a series, you cannot enter ∞ as the upper limit of summation. Can you still find the sum using a graphing utility? If so, which partial sum will result in 10, the exact sum of the series?

Application

Example 9 Increasing Annuity

A deposit of \$50 is made on the first day of each month in a savings account that pays 6% compounded monthly. What is the balance at the end of 2 years? (This type of savings plan is called an **increasing annuity.**)

Solution

The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1 + \frac{0.06}{12}\right)^{24} = 50(1.005)^{24}.$$

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1 + \frac{0.06}{12}\right)^{23} = 50(1.005)^{23}.$$

The last deposit will gain interest for only 1 month, and its balance will be

$$A_1 = 50\left(1 + \frac{0.06}{12}\right)^{1} = 50(1.005).$$

The total balance in the annuity will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.005)$ and $r = 1.005$, you have

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right) \qquad \text{Formula for sum of a finite geometric sequence}$$

$$S_{24} = 50(1.005)\left[\frac{1 - (1.005)^{24}}{1 - 1.005}\right] \qquad \text{Substitute } 50(1.005) \text{ for } a_1, 1.005 \text{ for } r, \text{ and } 24 \text{ for } n.$$

$$\approx \$1277.96. \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 85.

STUDY TIP

Recall from Section 3.1 that the compound interest formula is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

So, in Example 9, \$50 is the principal, 0.06 is the interest rate, 12 is the number of compoundings per year, and 2 is the time in years. If you substitute these values, you obtain

$$\begin{aligned} A &= 50\left(1 + \frac{0.06}{12}\right)^{12(2)} \\ &= 50\left(1 + \frac{0.06}{12}\right)^{24} \end{aligned}$$

8.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. A sequence is called a _______ sequence if the ratios of consecutive terms are the same. This ratio is called the _______ ratio.
2. The nth term of a geometric sequence has the form _______ .
3. The formula for the sum of a finite geometric sequence is given by _______ .
4. The sum of the terms of an infinite geometric sequence is called a _______ .
5. The formula for the sum of an infinite geometric series is given by _______ .

In Exercises 1–10, determine whether or not the sequence is geometric. If it is, find the common ratio.

1. $5, 15, 45, 135, \ldots$ **2.** $3, 12, 48, 192, \ldots$

3. $6, 18, 30, 42, \ldots$ **4.** $1, -2, 4, -8, \ldots$

5. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$ **6.** $5, 1, 0.2, 0.04, \ldots$

7. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \ldots$ **8.** $9, -6, 4, -\frac{8}{3}, \ldots$

9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ **10.** $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \ldots$

In Exercises 11–18, write the first five terms of the geometric sequence.

11. $a_1 = 6, \quad r = 3$ **12.** $a_1 = 4, \quad r = 2$

13. $a_1 = 1, \quad r = \frac{1}{2}$ **14.** $a_1 = 2, \quad r = \frac{1}{3}$

15. $a_1 = 5, \quad r = -\frac{1}{10}$ **16.** $a_1 = 6, \quad r = -\frac{1}{4}$

17. $a_i = 1, \quad r = e$ **18.** $a_1 = 4, \quad r = \sqrt{3}$

In Exercises 19–24, write the first five terms of the geometric sequence. Find the common ratio and write the *n*th term of the sequence as a function of *n*.

19. $a_1 = 64, \; a_{k+1} = \frac{1}{2}a_k$ **20.** $a_1 = 81, \; a_{k+1} = \frac{1}{3}a_k$

21. $a_1 = 9, \; a_{k+1} = 2a_k$ **22.** $a_1 = 5, \; a_{k+1} = -3a_k$

23. $a_1 = 6, \; a_{k+1} = -\frac{3}{2}a_k$ **24.** $a_1 = 30, \; a_{k+1} = -\frac{2}{3}a_k$

In Exercises 25–32, find the *n*th term of the geometric sequence. Use the *table* feature of a graphing utility to verify your answer numerically.

25. $a_1 = 4, \; a_4 = \frac{1}{2}, \; n = 10$ **26.** $a_1 = 5, \; a_3 = \frac{45}{4}, \; n = 8$

27. $a_1 = 6, \; r = -\frac{1}{3}, \; n = 12$ **28.** $a_1 = 8, \; r = -\frac{3}{4}, \; n = 9$

29. $a_1 = 500, \; r = 1.02, \; n = 14$

30. $a_1 = 1000, \; r = 1.005, \; n = 11$

31. $a_2 = -18, \; a_5 = \frac{2}{3}, \; n = 6$

32. $a_3 = \frac{16}{3}, \; a_5 = \frac{64}{27}, \; n = 7$

In Exercises 33–36, find a formula for the *n*th term of the geometric sequence. Then find the indicated *n*th term of the geometric sequence.

33. 9th term: $7, 21, 63, \ldots$

34. 7th term: $3, 36, 432, \ldots$

35. 10th term: $5, 30, 180, \ldots$

36. 22nd term: $4, 8, 16, \ldots$

In Exercises 37–40, use a graphing utility to graph the first 10 terms of the sequence.

37. $a_n = 12(-0.75)^{n-1}$

38. $a_n = 20(1.25)^{n-1}$

39. $a_n = 2(1.3)^{n-1}$

40. $a_n = 10(-1.2)^{n-1}$

In Exercises 41 and 42, find the first four terms of the sequence of partial sums of the geometric series. In a sequence of partial sums, the term S_n is the sum of the first *n* terms of the sequence. For instance, S_2 is the sum of the first two terms.

41. $8, -4, 2, -1, \frac{1}{2}, \ldots$

42. $8, 12, 18, 27, \frac{81}{2}, \ldots$

In Exercises 43 and 44, use a graphing utility to create a table showing the sequence of partial sums for the first 10 terms of the series.

43. $\sum_{n=1}^{\infty} 16\left(\frac{1}{2}\right)^{n-1}$ **44.** $\sum_{n=1}^{\infty} 4(0.2)^{n-1}$

In Exercises 45–54, find the sum. Use a graphing utility to verify your result.

45. $\sum_{n=1}^{9} 2^{n-1}$ **46.** $\sum_{n=1}^{9} (-2)^{n-1}$

47. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1}$

48. $\sum_{i=1}^{6} 32\left(\frac{1}{4}\right)^{i-1}$

49. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^{n}$

50. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^{n}$

51. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1}$

52. $\sum_{i=1}^{10} 5\left(-\frac{1}{3}\right)^{i-1}$

53. $\sum_{n=0}^{5} 300(1.06)^{n}$

54. $\sum_{n=0}^{6} 500(1.04)^{n}$

In Exercises 55–58, use summation notation to write the sum.

55. $5 + 15 + 45 + \cdots + 3645$

56. $7 + 14 + 28 + \cdots + 896$

57. $2 - \frac{1}{2} + \frac{1}{8} - \cdots + \frac{1}{2048}$

58. $15 - 3 + \frac{3}{5} - \cdots - \frac{3}{625}$

In Exercises 59–72, find the sum of the infinite geometric series, if possible. If not possible, explain why.

59. $\sum_{n=0}^{\infty} 10\left(\frac{4}{5}\right)^{n}$

60. $\sum_{n=0}^{\infty} 6\left(\frac{2}{3}\right)^{n}$

61. $\sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^{n}$

62. $\sum_{n=0}^{\infty} 9\left(-\frac{2}{3}\right)^{n}$

63. $\sum_{n=1}^{\infty} 2\left(\frac{7}{3}\right)^{n-1}$

64. $\sum_{n=1}^{\infty} 8\left(\frac{5}{3}\right)^{n-1}$

65. $\sum_{n=0}^{\infty} 10(0.11)^{n}$

66. $\sum_{n=0}^{\infty} 5(0.45)^{n}$

67. $\sum_{n=0}^{\infty} -3(-0.9)^{n}$

68. $\sum_{n=0}^{\infty} -10(-0.2)^{n}$

69. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$

70. $9 + 6 + 4 + \frac{8}{3} + \cdots$

71. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$

72. $-6 + 5 - \frac{25}{6} + \frac{125}{36} - \cdots$

In Exercises 73–76, find the rational number representation of the repeating decimal.

73. $0.\overline{36}$

74. $0.\overline{297}$

75. $1.2\overline{5}$

76. $1.3\overline{8}$

77. ***Compound Interest*** A principal of \$1000 is invested at 3% interest. Find the amount after 10 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

78. ***Compound Interest*** A principal of \$2500 is invested at 4% interest. Find the amount after 20 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

79. ***Annuity*** A deposit of \$100 is made at the beginning of each month in an account that pays 3% interest, compounded monthly. The balance A in the account at the end of 5 years is given by

$$A = 100\left(1 + \frac{0.03}{12}\right)^{1} + \cdots + 100\left(1 + \frac{0.03}{12}\right)^{60}.$$

Find A.

80. ***Annuity*** A deposit of \$50 is made at the beginning of each month in an account that pays 2% interest, compounded monthly. The balance A in the account at the end of 5 years is given by

$$A = 50\left(1 + \frac{0.02}{12}\right)^{1} + \cdots + 50\left(1 + \frac{0.02}{12}\right)^{60}.$$

Find A.

81. ***Annuity*** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded monthly. The balance A after t years is given by

$$A = P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^{2} + \cdots + P\left(1 + \frac{r}{12}\right)^{12t}.$$

Show that the balance is given by

$$A = P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right).$$

82. ***Annuity*** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded continuously. The balance A after t years is given by $A = Pe^{r/12} + Pe^{2r/12} + \cdots + Pe^{12tr/12}$. Show that the balance is given by

$$A = \frac{Pe^{r/12}(e^{rt} - 1)}{e^{r/12} - 1}.$$

Annuities **In Exercises 83–86, consider making monthly deposits of P dollars in a savings account earning an annual interest rate r. Use the results of Exercises 81 and 82 to find the balances A after t years if the interest is compounded (a) monthly and (b) continuously.**

83. $P = \$50,\ r = 7\%,\ t = 20$ years

84. $P = \$75,\ r = 4\%,\ t = 25$ years

85. $P = \$100,\ r = 5\%,\ t = 40$ years

86. $P = \$20,\ r = 6\%,\ t = 50$ years

87. ***Geometry*** The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the resulting triangles are shaded (see figure on the next page). If this process is repeated five more times, determine the total area of the shaded region.

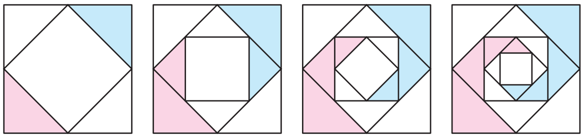
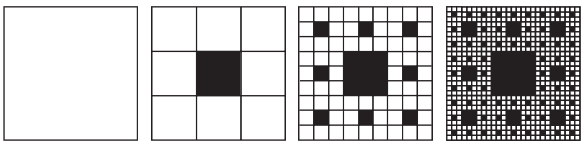


Figure for 87

88. ***Geometry*** The sides of a square are 27 inches in length. New squares are formed by dividing the original square into nine squares. The center square is then shaded (see figure). If this process is repeated three more times, determine the total area of the shaded region.



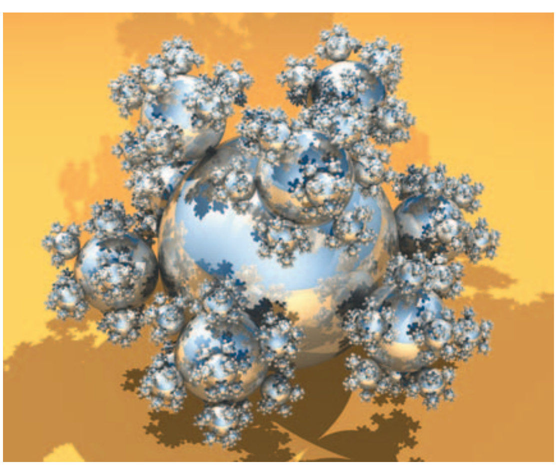
89. ***Temperature*** The temperature of water in an ice cube tray is 70°F when it is placed in the freezer. Its temperature n hours after being placed in the freezer is 20% less than 1 hour earlier.

(a) Find a formula for the nth term of the geometric sequence that gives the temperature of the water n hours after it is placed in the freezer.

(b) Find the temperatures of the water 6 hours and 12 hours after it is placed in the freezer.

(c) Use a graphing utility to graph the sequence to approximate the time required for the water to freeze.

90. ***Sphereflake*** The sphereflake shown is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. Attached to the large sphere are nine spheres of radius $\frac{1}{3}$. Attached to each of the smaller spheres are nine spheres of radius $\frac{1}{9}$. This process is continued infinitely.



Eric Haines

(a) Write a formula in series notation that gives the surface area of the sphereflake.

(b) Write a formula in series notation that gives the volume of the sphereflake.

(c) Determine if either the surface area or the volume of the sphereflake is finite or infinite. If either is finite, find the value.

Multiplier Effect **In Exercises 91–96, use the following information. A tax rebate is given to property owners by the state government with the anticipation that each property owner will spend approximately *p%* of the rebate, and in turn each recipient of this amount will spend *p%* of what they receive, and so on. Economists refer to this exchange of money and its circulation within the economy as the "multiplier effect." The multiplier effect operates on the principle that one individual's expenditure is another individual's income. Find the total amount put back into the state's economy, if this effect continues without end.**

	Tax rebate	*p%*
91.	$400	75%
92.	$500	70%
93.	$250	80%
94.	$350	75%
95.	$600	72.5%
96.	$450	77.5%

97. ***Salary Options*** You have been hired at a company and the administration offers you two salary options.

Option 1: a starting salary of $30,000 for the first year with salary increases of 2.5% per year for four years and then a reevaluation of performance

Option 2: a starting salary of $32,500 for the first year with salary increases of 2% per year for four years and then a reevaluation of performance

(a) Which option do you choose if you want to make the greater cumulative amount for the five-year period? Explain your reasoning.

(b) Which option do you choose if you want to make the greater amount the year prior to reevaluation? Explain your reasoning.

98. ***Manufacturing*** An electronics game manufacturer producing a new product estimates the annual sales to be 8000 units. Each year, 10% of the units that have been sold will become inoperative. So, 8000 units will be in use after 1 year, $[8000 + 0.9(8000)]$ units will be in use after 2 years, and so on.

(a) Write a formula in series notation for the number of units that will be operative after n years.

(b) Find the numbers of units that will be operative after 10 years, 20 years, and 50 years.

(c) If this trend continues indefinitely, will the number of units that will be operative be finite? If so, how many? If not, explain your reasoning.

99. ***Distance*** A bungee jumper is jumping off the New River Gorge Bridge in West Virginia, which has a height of 876 feet. The cord stretches 850 feet and the jumper rebounds 75% of the distance fallen.

(a) After jumping and rebounding 10 times, how far has the jumper traveled downward? How far has the jumper traveled upward? What is the total distance traveled downward and upward?

(b) Approximate the total distance both downward and upward, that the jumper travels before coming to rest.

100. ***Distance*** A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet.

(a) Find the total vertical distance traveled by the ball.

(b) The ball takes the following times (in seconds) for each fall.

$$\begin{aligned} s_1 &= -16t^2 + 16, & s_1 &= 0 \text{ if } t = 1\\ s_2 &= -16t^2 + 16(0.81), & s_2 &= 0 \text{ if } t = 0.9\\ s_3 &= -16t^2 + 16(0.81)^2, & s_3 &= 0 \text{ if } t = (0.9)^2\\ s_4 &= -16t^2 + 16(0.81)^3, & s_4 &= 0 \text{ if } t = (0.9)^3\\ &\vdots & &\vdots\\ s_n &= -16t^2 + 16(0.81)^{n-1}, & s_n &= 0 \text{ if } t = (0.9)^{n-1} \end{aligned}$$

Beginning with s_2, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is

$$t = 1 + 2\sum_{n=1}^{\infty}(0.9)^n.$$

Find this total time.

Synthesis

True or False? **In Exercises 101 and 102, determine whether the statement is true or false. Justify your answer.**

101. A sequence is geometric if the ratios of consecutive differences of consecutive terms are the same.

102. You can find the nth term of a geometric sequence by multiplying its common ratio by the first term of the sequence raised to the $(n - 1)$th power.

In Exercises 103 and 104, write the first five terms of the geometric sequence.

103. $a_1 = 3,\ r = \dfrac{x}{2}$

104. $a_1 = \dfrac{1}{2},\ r = 7x$

In Exercises 105 and 106, find the nth term of the geometric sequence.

105. $a_1 = 100,\ r = e^x,\ n = 9$

106. $a_1 = 4,\ r = \dfrac{4x}{3},\ n = 6$

107. ***Graphical Reasoning*** Use a graphing utility to graph each function. Identify the horizontal asymptote of the graph and determine its relationship to the sum.

(a) $f(x) = 6\left[\dfrac{1 - (0.5)^x}{1 - (0.5)}\right], \quad \displaystyle\sum_{n=0}^{\infty} 6\left(\frac{1}{2}\right)^n$

(b) $f(x) = 2\left[\dfrac{1 - (0.8)^x}{1 - (0.8)}\right], \quad \displaystyle\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$

108. ***Writing*** Write a brief paragraph explaining why the terms of a geometric sequence decrease in magnitude when $-1 < r < 1$.

109. ***Writing*** Write a brief paragraph explaining how to use the first two terms of a geometric sequence to find the nth term.

110. ***Exploration*** The terms of a geometric sequence can be written as

$$a_1,\ a_2 = a_1r,\ a_3 = a_2r,\ a_4 = a_3r,\ . \ . \ . \ .$$

Write each term of the sequence in terms of a_1 and r. Then based on the pattern, write the nth term of the geometric sequence.

Skills Review

111. ***Average Speed*** A truck traveled at an average speed of 50 miles per hour on a 200-mile trip. On the return trip, the average speed was 42 miles per hour. Find the average speed for the round trip.

112. ***Work Rate*** Your friend can mow a lawn in 4 hours and you can mow it in 6 hours. How long will it take both of you to mow the lawn working together?

In Exercises 113 and 114, find the determinant of the matrix.

113. $\begin{bmatrix} -1 & 3 & 4\\ -2 & 8 & 0\\ 2 & 5 & -1 \end{bmatrix}$

114. $\begin{bmatrix} -1 & 0 & 4\\ -4 & 3 & 5\\ 0 & 2 & -3 \end{bmatrix}$

115. **Make a Decision** To work an extended application analyzing the monthly profits of a clothing manufacturer over a period of 36 months, visit this textbook's *Online Study Center.*

8.4 Mathematical Induction

Introduction

In this section, you will study a form of mathematical proof called **mathematical induction.** It is important that you clearly see the logical need for it, so let's take a closer look at the problem discussed in Example 5(a) on page 595.

$$S_1 = 1 = 1^2$$

$$S_2 = 1 + 3 = 2^2$$

$$S_3 = 1 + 3 + 5 = 3^2$$

$$S_4 = 1 + 3 + 5 + 7 = 4^2$$

$$S_5 = 1 + 3 + 5 + 7 + 9 = 5^2$$

Judging from the pattern formed by these first five sums, it appears that the sum of the first n odd integers is

$$S_n = 1 + 3 + 5 + 7 + 9 + \cdot \cdot \cdot + (2n - 1) = n^2.$$

Although this particular formula is valid, it is important for you to see that recognizing a pattern and then simply *jumping to the conclusio*n that the pattern must be true for all values of n is *not* a logically valid method of proof. There are many examples in which a pattern appears to be developing for small values of n but then fails at some point. One of the most famous cases of this is the conjecture by the French mathematician Pierre de Fermat (1601–1665), who speculated that all numbers of the form

$$F_n = 2^{2^n} + 1, \qquad n = 0, 1, 2, . . .$$

are prime. For $n = 0, 1, 2, 3$, and 4, the conjecture is true.

$$F_0 = 3$$

$$F_1 = 5$$

$$F_2 = 17$$

$$F_3 = 257$$

$$F_4 = 65,537$$

The size of the next *Fermat number* ($F_5 = 4,294,967,297$) is so great that it was difficult for Fermat to determine whether or not it was prime. However, another well-known mathematician, Leonhard Euler (1707–1783), later found a factorization

$$F_5 = 4,294,967,297$$
$$= 641(6,700,417)$$

which proved that F_5 is not prime and therefore Fermat's conjecture was false.

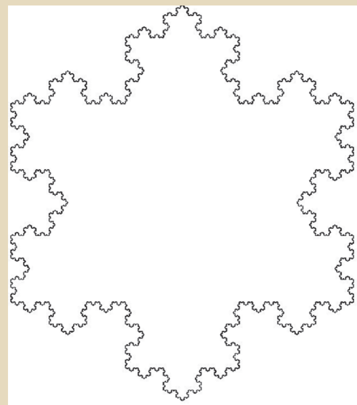
Just because a rule, pattern, or formula seems to work for several values of n, you cannot simply decide that it is valid for *all* values of n without going through a *legitimate proof*. Mathematical induction is one method of proof.

What you should learn

- Use mathematical induction to prove statements involving a positive integer n.
- Find the sums of powers of integers.
- Find finite differences of sequences.

Why you should learn it

Finite differences can be used to determine what type of model can be used to represent a sequence. For instance, in Exercise 55 on page 618, you will use finite differences to find a model that represents the number of sides of the nth Koch snowflake.



Courtesy of Stefan Steinhaus

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If

1. P_1 is true, and
2. the truth of P_k implies the truth of P_{k+1} for every positive integer k,

then P_n must be true for all positive integers n.

STUDY TIP

It is important to recognize that in order to prove a statement by induction, *both* parts of the Principle of Mathematical Induction are necessary.

To apply the Principle of Mathematical Induction, you need to be able to determine the statement P_{k+1} for a given statement P_k. To determine P_{k+1}, substitute $k + 1$ for k in the statement P_k.

Example 1 A Preliminary Example

Find P_{k+1} for each P_k.

a. $P_k : S_k = \dfrac{k^2(k+1)^2}{4}$

b. $P_k : S_k = 1 + 5 + 9 + \cdot \cdot \cdot + [4(k-1) - 3] + (4k - 3)$

c. $P_k : k + 3 < 5k^2$

d. $P_k : 3^k \geq 2k + 1$

Solution

a. $P_{k+1} : S_{k+1} = \dfrac{(k+1)^2(k+1+1)^2}{4}$ Replace k by $k + 1$.

$= \dfrac{(k+1)^2(k+2)^2}{4}$ Simplify.

b. $P_{k+1} : S_{k+1} = 1 + 5 + 9 + \cdot \cdot \cdot + \{4[(k+1) - 1] - 3\} + [4(k+1) - 3]$

$= 1 + 5 + 9 + \cdot \cdot \cdot + (4k - 3) + (4k + 1)$

c. $P_{k+1} : (k+1) + 3 < 5(k+1)^2$

$k + 4 < 5(k^2 + 2k + 1)$

d. $P_{k+1} : 3^{k+1} \geq 2(k+1) + 1$

$3^{k+1} \geq 2k + 3$

CHECKPOINT Now try Exercise 5.

A well-known illustration used to explain why the Principle of Mathematical Induction works is the unending line of dominoes represented by Figure 8.15. If the line actually contains infinitely many dominoes, it is clear that you could not knock down the entire line by knocking down only *one domino* at a time. However, suppose it were true that each domino would knock down the next one as it fell. Then you could knock them all down simply by pushing the first one and starting a chain reaction. Mathematical induction works in the same way. If the truth of P_k implies the truth of P_{k+1} and if P_1 is true, the chain reaction proceeds as follows: P_1 implies P_2, P_2 implies P_3, P_3 implies P_4, and so on.

Figure 8.15

When using mathematical induction to prove a *summation* formula (such as the one in Example 2), it is helpful to think of S_{k+1} as

$$S_{k+1} = S_k + a_{k+1}$$

where a_{k+1} is the $(k + 1)$th term of the original sum.

Example 2 Using Mathematical Induction

Use mathematical induction to prove the following formula.

$$S_n = 1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$$

Solution

Mathematical induction consists of two distinct parts. First, you must show that the formula is true when $n = 1$.

1. When $n = 1$, the formula is valid because

$$S_1 = 1 = 1^2.$$

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for *some* integer k. The second step is to use this assumption to prove that the formula is valid for the next integer, $k + 1$.

2. Assuming that the formula

$$S_k = 1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \cdots + (2k - 1) + [2(k + 1) - 1] \\ &= [1 + 3 + 5 + 7 + \cdots + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + (2k + 1) && \text{Group terms to form } S_k. \\ &= k^2 + 2k + 1 && \text{Replace } S_k \text{ by } k^2. \\ &= (k + 1)^2 \end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for all positive integer values of n.

CHECKPOINT Now try Exercise 7.

It occasionally happens that a statement involving natural numbers is *not* true for the first $k - 1$ positive integers but *is* true for all values of $n \geq k$. In these instances, you use a slight variation of the Principle of Mathematical Induction in which you verify P_k rather than P_1. This variation is called the *Extended Principle of Mathematical Induction*. To see the validity of this principle, note from Figure 8.15 that all but the first $k - 1$ dominoes can be knocked down by knocking over the kth domino. This suggests that you can prove a statement P_n to be true for $n \geq k$ by showing that P_k is true and that P_k implies P_{k+1}. In Exercises 25–30 in this section, you are asked to apply this extension of mathematical induction.

Example 3 Using Mathematical Induction

Use mathematical induction to prove the formula

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 1$.

Solution

1. When $n = 1$, the formula is valid because

$$S_1 = 1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1(2)(3)}{6}.$$

2. Assuming that

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + k^2 = \frac{k(k+1)(2k+1)}{6}$$

you must show that

$$S_{k+1} = \frac{(k+1)(k+1+1)[2(k+1)+1]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.$$

To do this, write the following.

$$\begin{aligned}
S_{k+1} &= S_k + a_{k+1} \\
&= (1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + k^2) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{By assumption} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for *all* integers $n \geq 1$.

CHECKPOINT Now try Exercise 13.

When proving a formula by mathematical induction, the only statement that you *need* to verify is P_1. As a check, it is a good idea to try verifying some of the other statements. For instance, in Example 3, try verifying P_2 and P_3.

Sums of Powers of Integers

The formula in Example 3 is one of a collection of useful summation formulas. This and other formulas dealing with the sums of various powers of the first n positive integers are summarized below.

Sums of Powers of Integers

1. $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \cdot \cdot \cdot + n = \frac{n(n+1)}{2}$

2. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdot \cdot \cdot + n^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdot \cdot \cdot + n^3 = \frac{n^2(n+1)^2}{4}$

4. $\sum_{i=1}^{n} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + \cdot \cdot \cdot + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

5. $\sum_{i=1}^{n} i^5 = 1^5 + 2^5 + 3^5 + 4^5 + \cdot \cdot \cdot + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

Each of these formulas for sums can be proven by mathematical induction. (See Exercises 13–16 in this section.)

Example 4 Proving an Inequality by Mathematical Induction

Prove that $n < 2^n$ for all integers $n \geq 1$.

Solution

1. For $n = 1$ and $n = 2$, the formula is true because

$$1 < 2^1 \text{ and } 2 < 2^2.$$

2. Assuming that

$$k < 2^k$$

you need to show that $k + 1 < 2^{k+1}$. Multiply each side of $k < 2^k$ by 2.

$$2(k) < 2(2^k) = 2^{k+1}$$

Because $k + 1 < k + k = 2k$ for all $k > 1$, it follows that

$$k + 1 < 2k < 2^{k+1}$$

or

$$k + 1 < 2^{k+1}.$$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that $n < 2^n$ for all integers $n \geq 1$.

CHECKPOINT Now try Exercise 25.

Finite Differences

The **first differences** of a sequence are found by subtracting consecutive terms. The **second differences** are found by subtracting consecutive first differences. The first and second differences of the sequence 3, 5, 8, 12, 17, 23, . . . are as follows.

n:	1		2		3		4		5		6
a_n:	3		5		8		12		17		23
First differences:		2		3		4		5		6	
Second differences:			1		1		1		1		

For this sequence, the second differences are all the same. When this happens, and the second differences are nonzero, the sequence has a perfect *quadratic* model. If the first differences are all the same nonzero number, the sequence has a *linear* model—that is, it is arithmetic.

STUDY TIP

For a linear model, the *first* differences are the same nonzero number. For a quadratic model, the *second* differences are the same nonzero number.

Example 5 Finding a Quadratic Model

Find the quadratic model for the sequence 3, 5, 8, 12, 17, 23, ...

Solution

You know from the second differences shown above that the model is quadratic and has the form

$$a_n = an^2 + bn + c.$$

By substituting 1, 2, and 3 for n, you can obtain a system of three linear equations in three variables.

$$a_1 = a(1)^2 + b(1) + c = 3 \qquad \text{Substitute 1 for } n.$$
$$a_2 = a(2)^2 + b(2) + c = 5 \qquad \text{Substitute 2 for } n.$$
$$a_3 = a(3)^2 + b(3) + c = 8 \qquad \text{Substitute 3 for } n.$$

Prerequisite Skills

If you have difficulty in solving the system of equations in this example, review Gaussian elimination in Section 7.3.

You now have a system of three equations in a, b, and c.

$$\begin{cases} a + b + c = 3 & \text{Equation 1} \\ 4a + 2b + c = 5 & \text{Equation 2} \\ 9a + 3b + c = 8 & \text{Equation 3} \end{cases}$$

Solving this system of equations using the techniques discussed in Chapter 7, you can find the solution to be $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 2$. So, the quadratic model is

$$a_n = \tfrac{1}{2}n^2 + \tfrac{1}{2}n + 2.$$

Check the values of a_1, a_2, and a_3 as follows.

Check

$$a_1 = \tfrac{1}{2}(1)^2 + \tfrac{1}{2}(1) + 2 = 3 \qquad \text{Solution checks. ✓}$$
$$a_2 = \tfrac{1}{2}(2)^2 + \tfrac{1}{2}(2) + 2 = 5 \qquad \text{Solution checks. ✓}$$
$$a_3 = \tfrac{1}{2}(3)^2 + \tfrac{1}{2}(3) + 2 = 8 \qquad \text{Solution checks. ✓}$$

CHECKPOINT Now try Exercise 51.

8.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The first step in proving a formula by _______ is to show that the formula is true when $n = 1$.
2. The _______ differences of a sequence are found by subtracting consecutive terms.
3. A sequence is an _______ sequence if the first differences are all the same nonzero number.
4. If the _______ differences of a sequence are all the same nonzero number, then the sequence has a perfect quadratic model.

In Exercises 1–6, find P_{k+1} for the given P_k.

1. $P_k = \dfrac{5}{k(k+1)}$ **2.** $P_k = \dfrac{4}{(k+2)(k+3)}$

3. $P_k = \dfrac{2^k}{(k+1)!}$ **4.** $P_k = \dfrac{2^{k-1}}{k!}$

5. $P_k = 1 + 6 + 11 + \cdot \cdot \cdot + [5(k-1) - 4] + (5k - 4)$

6. $P_k = 7 + 13 + 19 + \cdot \cdot \cdot + [6(k-1) + 1] + (6k + 1)$

In Exercises 7–20, use mathematical induction to prove the formula for every positive integer n.

7. $2 + 4 + 6 + 8 + \cdot \cdot \cdot + 2n = n(n+1)$

8. $3 + 11 + 19 + 27 + \cdot \cdot \cdot + (8n - 5) = n(4n - 1)$

9. $3 + 8 + 13 + 18 + \cdot \cdot \cdot + (5n - 2) = \dfrac{n}{2}(5n + 1)$

10. $1 + 4 + 7 + 10 + \cdot \cdot \cdot + (3n - 2) = \dfrac{n}{2}(3n - 1)$

11. $1 + 2 + 2^2 + 2^3 + \cdot \cdot \cdot + 2^{n-1} = 2^n - 1$

12. $2(1 + 3 + 3^2 + 3^3 + \cdot \cdot \cdot + 3^{n-1}) = 3^n - 1$

13. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

14. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

15. $\displaystyle\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

16. $\displaystyle\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

17. $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

18. $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

19. $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

20. $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

In Exercises 21–24, find the sum using the formulas for the sums of powers of integers.

21. $\displaystyle\sum_{n=1}^{50} n^3$ **22.** $\displaystyle\sum_{n=1}^{10} n^4$

23. $\displaystyle\sum_{n=1}^{12} (n^2 - n)$

24. $\displaystyle\sum_{n=1}^{40} (n^3 - n)$

In Exercises 25–30, prove the inequality for the indicated integer values of n.

25. $n! > 2^n, \quad n \geq 4$

26. $\left(\frac{4}{3}\right)^n > n, \quad n \geq 7$

27. $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdot \cdot \cdot + \dfrac{1}{\sqrt{n}} > \sqrt{n}, \quad n \geq 2$

28. $\left(\dfrac{x}{y}\right)^{n+1} < \left(\dfrac{x}{y}\right)^n, \quad n \geq 1$ and $0 < x < y$

29. $(1 + a)^n \geq na, \quad n \geq 1$ and $a > 1$

30. $3^n > n\,2^n, \quad n \geq 1$

In Exercises 31–42, use mathematical induction to prove the property for all positive integers n.

31. $(ab)^n = a^n b^n$

32. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

33. If $x_1 \neq 0,\ x_2 \neq 0, \ldots, x_n \neq 0$, then
$(x_1 x_2 x_3 \cdot \cdot \cdot x_n)^{-1} = x_1^{-1} x_2^{-1} x_3^{-1} \cdot \cdot \cdot x_n^{-1}$.

34. If $x_1 > 0,\ x_2 > 0, \ldots, x_n > 0$, then
$\ln(x_1 x_2 \cdot \cdot \cdot x_n) = \ln x_1 + \ln x_2 + \cdot \cdot \cdot + \ln x_n$.

35. Generalized Distributive Law:
$x(y_1 + y_2 + \cdot \cdot \cdot + y_n) = xy_1 + xy_2 + \cdot \cdot \cdot + xy_n$

36. $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for all $n \geq 1$.

37. A factor of $(n^3 + 3n^2 + 2n)$ is 3.

38. A factor of $(n^3 + 5n + 6)$ is 3.

39. A factor of $(n^3 - n + 3)$ is 3.

40. A factor of $(n^4 - n + 4)$ is 2.

41. A factor of $(2^{2n+1} + 1)$ is 3.

42. A factor of $(2^{4n-2} + 1)$ is 5.

In Exercises 43–50, write the first five terms of the sequence beginning with the given term. Then calculate the first and second differences of the sequence. Does the sequence have a linear model, a quadratic model, or neither?

43. $a_1 = 0$, $a_n = a_{n-1} + 3$

44. $a_1 = 2$, $a_n = n - a_{n-1}$

45. $a_1 = 3$, $a_n = a_{n-1} - n$

46. $a_2 = -3$, $a_n = -2a_{n-1}$

47. $a_0 = 0$, $a_n = a_{n-1} + n$

48. $a_0 = 2$, $a_n = (a_{n-1})^2$

49. $a_1 = 2$, $a_n = a_{n-1} + 2$

50. $a_1 = 0$, $a_n = a_{n-1} + 2n$

In Exercises 51–54, find a quadratic model for the sequence with the indicated terms.

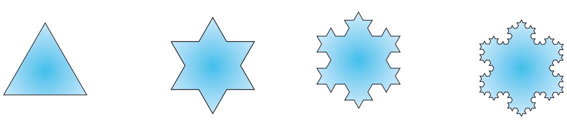
51. 3, 3, 5, 9, 15, 23, . . .

52. 7, 6, 7, 10, 15, 22, . . .

53. $a_0 = -3,\ a_2 = 1,\ a_4 = 9$

54. $a_0 = 3,\ a_2 = 0,\ a_6 = 36$

55. ***Koch Snowflake*** A *Koch snowflake* is created by starting with an equilateral triangle with sides one unit in length. Then, on each side of the triangle, a new equilateral triangle is created on the middle third of that side. This process is repeated continuously, as shown in the figure.

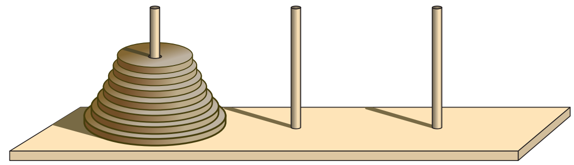


(a) Determine a formula for the number of sides of the nth Koch snowflake. Use mathematical induction to prove your answer.

(b) Determine a formula for the area of the nth Koch snowflake. Recall that the area A of an equilateral triangle with side s is $A = (\sqrt{3}/4)s^2$.

(c) Determine a formula for the perimeter of the nth Koch snowflake.

56. ***Tower of Hanoi*** The *Tower of Hanoi* puzzle is a game in which three pegs are attached to a board and one of the pegs has n disks sitting on it, as shown in the figure. Each disk on that peg must sit on a larger disk. The strategy of the game is to move the entire pile of disks, one at a time, to another peg. At no time may a disk sit on a smaller disk.



(a) Find the number of moves if there are three disks.

(b) Find the number of moves if there are four disks.

(c) Use your results from parts (a) and (b) to find a formula for the number of moves if there are n disks.

(d) Use mathematical induction to prove the formula you found in part (c).

Synthesis

True or False? **In Exercises 57–59, determine whether the statement is true or false. Justify your answer.**

57. If the statement P_k is true and P_k implies P_{k+1}, then P_1 is also true.

58. If a sequence is arithmetic, then the first differences of the sequence are all zero.

59. A sequence with n terms has $n - 1$ second differences.

60. ***Think About It*** What conclusion can be drawn from the given information about the sequence of statements P_n?

(a) P_3 is true and P_k implies P_{k+1}.

(b) $P_1, P_2, P_3, \ldots, P_{50}$ are all true.

(c) P_1, P_2, and P_3 are all true, but the truth of P_k does not imply that P_{k+1} is true.

(d) P_2 is true and P_{2k} implies P_{2k+2}.

Skills Review

In Exercises 61–64, find the product.

61. $(2x^2 - 1)^2$

62. $(2x - y)^2$

63. $(5 - 4x)^3$

64. $(2x - 4y)^3$

In Exercises 65–68, simplify the expression.

65. $3\sqrt{-27} - \sqrt{-12}$

66. $\sqrt[3]{125} + 4\sqrt[3]{-8} - 2\sqrt[3]{-54}$

67. $10\left(\sqrt[3]{64} - 2\sqrt[3]{-16}\right)$

68. $\left(-5 + \sqrt{-9}\right)^2$

8.5 The Binomial Theorem

Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that provides a quick method of raising a binomial to a power. To begin, look at the expansion of

$$(x+y)^n$$

for several values of n.

$$(x+y)^0 = 1$$

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.
2. In each expansion, x and y have symmetric roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.
3. The sum of the powers of each term is n. For instance, in the expansion of $(x+y)^5$, the sum of the powers of each term is 5.

$$(x+y)^5 = x^5 + 5\overbrace{x^4y^1}^{4+1=5} + 10\overbrace{x^3y^2}^{3+2=5} + 10x^2y^3 + 5x^1y^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.** To find them, you can use the **Binomial Theorem.**

What you should learn

- Use the Binomial Theorem to calculate binomial coefficients.
- Use binomial coefficients to write binomial expansions.
- Use Pascal's Triangle to calculate binomial coefficients.

Why you should learn it

You can use binomial coefficients to predict future behavior. For instance, in Exercise 106 on page 625, you are asked to use binomial coefficients to find the probability that a baseball player gets three hits during the next 10 times at bat.

Jonathan Daniel/Getty Images

The Binomial Theorem (See the proof on page 658.)

In the expansion of $(x+y)^n$

$$(x+y)^n = x^n + nx^{n-1}y + \cdot \cdot \cdot + {}_nC_r\, x^{n-r}y^r + \cdot \cdot \cdot + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote binomial coefficients.

Prerequisite Skills

Review the definition of factorial, $n!$, in Section 8.1.

Example 1 Finding Binomial Coefficients

Find each binomial coefficient.

a. ${}_8C_2$ **b.** $\binom{10}{3}$ **c.** ${}_7C_0$ **d.** $\binom{8}{8}$

Solution

a. $${}_8C_2 = \frac{8!}{6! \cdot 2!} = \frac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

b. $$\binom{10}{3} = \frac{10!}{7! \cdot 3!} = \frac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

c. $${}_7C_0 = \frac{\cancel{7!}}{\cancel{7!} \cdot 0!} = 1$$

d. $$\binom{8}{8} = \frac{\cancel{8!}}{0! \cdot \cancel{8!}} = 1$$

CHECKPOINT Now try Exercise 1.

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) of Example 1, there is a simple pattern for evaluating binomial coefficients that works because there will always be factorial terms that divide out from the expression.

$${}_8C_2 = \frac{\overbrace{8 \cdot 7}^{\text{2 factors}}}{\underbrace{2 \cdot 1}_{\text{2 factorial}}} \quad \text{and} \quad \binom{10}{3} = \frac{\overbrace{10 \cdot 9 \cdot 8}^{\text{3 factors}}}{\underbrace{3 \cdot 2 \cdot 1}_{\text{3 factorial}}}$$

Example 2 Finding Binomial Coefficients

Find each binomial coefficient using the pattern shown above.

a. ${}_7C_3$ **b.** ${}_7C_4$ **c.** ${}_{12}C_1$ **d.** ${}_{12}C_{11}$

Solution

a. $${}_7C_3 = \frac{7 \cdot \cancel{6} \cdot 5}{\cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

b. $${}_7C_4 = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 35$$

c. $${}_{12}C_1 = \frac{12}{1} = 12$$

d. $${}_{12}C_{11} = \frac{12!}{1! \cdot 11!} = \frac{(12) \cdot \cancel{11!}}{1! \cdot \cancel{11!}} = \frac{12}{1} = 12$$

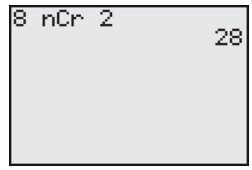
CHECKPOINT Now try Exercise 5.

It is not a coincidence that the results in parts (a) and (b) of Example 2 are the same and that the results in parts (c) and (d) are the same. In general, it is true that

$${}_nC_r = {}_nC_{n-r}.$$

TECHNOLOGY SUPPORT

Most graphing utilities are programmed to evaluate ${}_nC_r$. The figure below shows how one graphing utility evaluates the binomial coefficient in Example 1(a). For instructions on how to use the ${}_nC_r$ feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.



Exploration

Find each pair of binomial coefficients.

a. ${}_7C_0$, ${}_7C_7$ **d.** ${}_7C_1$, ${}_7C_6$

b. ${}_8C_0$, ${}_8C_8$ **e.** ${}_8C_1$, ${}_8C_7$

c. ${}_{10}C_0$, ${}_{10}C_{10}$ **f.** ${}_{10}C_1$, ${}_{10}C_9$

What do you observe about the pairs in (a), (b), and (c)? What do you observe about the pairs in (d), (e), and (f)? Write two conjectures from your observations. Develop a convincing argument for your two conjectures.

Binomial Expansions

As mentioned at the beginning of this section, when you write out the coefficients for a binomial that is raised to a power, you are **expanding a binomial.** The formulas for binomial coefficients give you an easy way to expand binomials, as demonstrated in the next four examples.

Example 3 Expanding a Binomial

Write the expansion of the expression $(x + 1)^3$.

Solution

The binomial coefficients are

$${}_3C_0 = 1,\ {}_3C_1 = 3,\ {}_3C_2 = 3, \text{ and } {}_3C_3 = 1.$$

Therefore, the expansion is as follows.

$$\begin{aligned}(x + 1)^3 &= (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)\\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

CHECKPOINT Now try Exercise 17.

To expand binomials representing *differences*, rather than sums, you alternate signs. Here is an example.

$$\begin{aligned}(x - 1)^3 &= [x + (-1)]^3\\ &= (1)x^3 + (3)x^2(-1) + (3)x(-1)^2 + (1)(-1)^3\\ &= x^3 - 3x^2 + 3x - 1\end{aligned}$$

Example 4 Expanding Binomial Expressions

Write the expansion of each expression.

a. $(2x - 3)^4$

b. $(x - 2y)^4$

Solution

The binomial coefficients are

$${}_4C_0 = 1,\ {}_4C_1 = 4,\ {}_4C_2 = 6,\ {}_4C_3 = 4, \text{ and } {}_4C_4 = 1.$$

Therefore, the expansions are as follows.

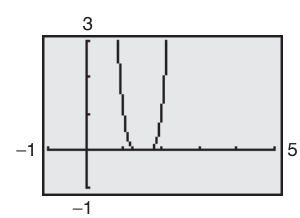
a. $$\begin{aligned}(2x - 3)^4 &= (1)(2x)^4 - (4)(2x)^3(3) + (6)(2x)^2(3^2) - (4)(2x)(3^3) + (1)(3^4)\\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$$

b. $$\begin{aligned}(x - 2y)^4 &= (1)x^4 - (4)x^3(2y) + (6)x^2(2y)^2 - (4)x(2y)^3 + (1)(2y)^4\\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4\end{aligned}$$

CHECKPOINT Now try Exercise 29.

TECHNOLOGY TIP

You can use a graphing utility to check the expansion in Example 4(a) by graphing the original binomial expression and the expansion in the same viewing window. The graphs should coincide, as shown below.



Example 5 Expanding a Binomial

Write the expansion of the expression $(x^2 + 4)^3$.

Solution

Expand using the binomial coefficients from Example 3.

$$(x^2 + 4)^3 = (1)(x^2)^3 + (3)(x^2)^2(4) + (3)x^2(4^2) + (1)(4^3)$$
$$= x^6 + 12x^4 + 48x^2 + 64$$

CHECKPOINT Now try Exercise 31.

Sometimes you will need to find a specific term in a binomial expansion. Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem, the $(r + 1)$st term is

$${}_nC_r x^{n-r} y^r.$$

For example, if you wanted to find the third term of the expression in Example 5, you could use the formula above with $n = 3$ and $r = 2$ to obtain

$${}_3C_2(x^2)^{3-2} \cdot 4^2 = 3(x^2) \cdot 16$$
$$= 48x^2.$$

Example 6 Finding a Term or Coefficient in a Binomial Expansion

a. Find the sixth term of $(a + 2b)^8$.

b. Find the coefficient of the term a^6b^5 in the expansion of $(2a - 5b)^{11}$.

Solution

a. To find the sixth term in this binomial expansion, use $n = 8$ and $r = 5$ [the formula is for the $(r + 1)$st term, so r is one less than the number of the term that you are looking for] to get

$${}_8C_5 a^{8-5}(2b)^5 = 56 \cdot a^3 \cdot (2b)^5$$
$$= 56(2^5)a^3b^5$$
$$= 1792a^3b^5.$$

b. In this case, $n = 11$, $r = 5$, $x = 2a$, and $y = -5b$. Substitute these values to obtain

$${}_nC_r x^{n-r} y^r = {}_{11}C_5(2a)^6(-5b)^5$$
$$= (462)(64a^6)(-3125b^5)$$
$$= -92{,}400{,}000a^6b^5.$$

So, the coefficient is $-92{,}400{,}000$.

CHECKPOINT Now try Exercises 49 and 61.

Pascal's Triangle

There is a convenient way to remember the pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle.** This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

$4 + 6 = 10$

$15 + 6 = 21$

The first and last number in each row of Pascal's Triangle is 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that the numbers in this triangle are precisely the same numbers as the coefficients of binomial expansions, as follows.

$$(x + y)^0 = 1 \qquad \text{0th row}$$
$$(x + y)^1 = 1x + 1y \qquad \text{1st row}$$
$$(x + y)^2 = 1x^2 + 2xy + 1y^2 \qquad \text{2nd row}$$
$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \qquad \text{3rd row}$$
$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \qquad \vdots$$
$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$
$$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$
$$(x + y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

The top row of Pascal's Triangle is called the *zeroth row* because it corresponds to the binomial expansion $(x + y)^0 = 1$. Similarly, the next row is called the *first row* because it corresponds to the binomial expansion $(x + y)^1 = 1(x) + 1(y)$. In general, the *nth row* of Pascal's Triangle gives the coefficients of $(x + y)^n$.

Example 7 Using Pascal's Triangle

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

$${}_8C_0,\ {}_8C_1,\ {}_8C_2,\ {}_8C_3,\ {}_8C_4,\ {}_8C_5,\ {}_8C_6,\ {}_8C_7,\ {}_8C_8$$

Solution

$$\begin{array}{ccccccccccccccccc}
&1&&7&&21&&35&&35&&21&&7&&1& \leftarrow \text{Seventh row} \\
1&&8&&28&&56&&70&&56&&28&&8&&1 \\
{}_8C_0&&{}_8C_1&&{}_8C_2&&{}_8C_3&&{}_8C_4&&{}_8C_5&&{}_8C_6&&{}_8C_7&&{}_8C_8
\end{array}$$

CHECKPOINT Now try Exercise 65.

Exploration

Complete the table and describe the result.

n	r	${}_nC_r$	${}_nC_{n-r}$
9	5		
7	1		
12	4		
6	0		
10	7		

What characteristics of Pascal's Triangle are illustrated by this table?

8.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The coefficients of a binomial expansion are called _______ .
2. To find binomial coefficients you can use the _______ or _______ .
3. The notation used to denote a binomial coefficient is _______ or _______ .
4. When you write out the coefficients for a binomial that is raised to a power, you are _______ a _______ .

In Exercises 1–10, find the binomial coefficient.

1. ${}_7C_5$
2. ${}_9C_6$
3. $\binom{12}{0}$
4. $\binom{20}{20}$
5. ${}_{20}C_{15}$
6. ${}_{12}C_3$
7. ${}_{14}C_1$
8. ${}_{18}C_{17}$
9. $\binom{100}{98}$
10. $\binom{10}{7}$

In Exercises 11–16, use a graphing utility to find ${}_nC_r$.

11. ${}_{41}C_{36}$
12. ${}_{34}C_4$
13. ${}_{100}C_{98}$
14. ${}_{500}C_{498}$
15. ${}_{250}C_2$
16. ${}_{1000}C_2$

In Exercises 17–48, use the Binomial Theorem to expand and simplify the expression.

17. $(x + 2)^4$
18. $(x + 1)^6$
19. $(a + 3)^3$
20. $(a + 2)^4$
21. $(y - 2)^4$
22. $(y - 2)^5$
23. $(x + y)^5$
24. $(x + y)^6$
25. $(3r + 2s)^6$
26. $(4x + 3y)^4$
27. $(x - y)^5$
28. $(2x - y)^5$
29. $(1 - 4x)^3$
30. $(5 - 2y)^3$
31. $(x^2 + 2)^4$
32. $(3 - y^2)^3$
33. $(x^2 - 5)^5$
34. $(y^2 + 1)^6$
35. $(x^2 + y^2)^4$
36. $(x^2 + y^2)^6$
37. $(x^3 - y)^6$
38. $(2x^3 - y)^5$
39. $\left(\frac{1}{x} + y\right)^5$
40. $\left(\frac{1}{x} + 2y\right)^6$
41. $\left(\frac{2}{x} - y\right)^4$
42. $\left(\frac{2}{x} - 3y\right)^5$
43. $(4x - 1)^3 - 2(4x - 1)^4$
44. $(x + 3)^5 - 4(x + 3)^4$
45. $2(x - 3)^4 + 5(x - 3)^2$
46. $3(x + 1)^5 + 4(x + 1)^3$
47. $-3(x - 2)^3 - 4(x + 1)^6$
48. $5(x + 2)^5 - 2(x - 1)^2$

In Exercises 49–56, find the specified nth term in the expansion of the binomial.

49. $(x + 8)^{10}$, $n = 4$
50. $(x - 5)^6$, $n = 7$
51. $(x - 6y)^5$, $n = 3$
52. $(x - 10z)^7$, $n = 4$
53. $(4x + 3y)^9$, $n = 8$
54. $(5a + 6b)^5$, $n = 5$
55. $(10x - 3y)^{12}$, $n = 9$
56. $(7x + 2y)^{15}$, $n = 8$

In Exercises 57–64, find the coefficient a of the given term in the expansion of the binomial.

	Binomial	*Term*
57.	$(x + 3)^{12}$	ax^4
58.	$(x + 4)^{12}$	ax^5
59.	$(x - 2y)^{10}$	ax^8y^2
60.	$(4x - y)^{10}$	ax^2y^8
61.	$(3x - 2y)^9$	ax^6y^3
62.	$(2x - 3y)^8$	ax^4y^4
63.	$(x^2 + y)^{10}$	ax^8y^6
64.	$(z^2 - 1)^{12}$	az^6

In Exercises 65–68, use Pascal's Triangle to find the binomial coefficient.

65. ${}_7C_5$
66. ${}_6C_3$
67. ${}_6C_5$
68. ${}_5C_2$

In Exercises 69–72, expand the binomial by using Pascal's Triangle to determine the coefficients.

69. $(3t - 2v)^4$
70. $(5v - 2z)^4$
71. $(2x - 3y)^5$
72. $(5y + 2)^5$

In Exercises 73–76, use the Binomial Theorem to expand and simplify the expression.

73. $(\sqrt{x} + 5)^4$
74. $(4\sqrt{t} - 1)^3$
75. $(x^{2/3} - y^{1/3})^3$
76. $(u^{3/5} + v^{1/5})^5$

In Exercises 77–82, expand the expression in the difference quotient and simplify.

$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0$$

77. $f(x) = x^3$
78. $f(x) = x^4$
79. $f(x) = x^6$
80. $f(x) = x^8$
81. $f(x) = \sqrt{x}$
82. $f(x) = \dfrac{1}{x}$

In Exercises 83–96, use the Binomial Theorem to expand the complex number. Simplify your result. (Remember that $i = \sqrt{-1}$.

83. $(1 + i)^4$
84. $(4 - i)^5$
85. $(4 + i)^4$
86. $(2 - i)^5$
87. $(2 - 3i)^6$
88. $(3 - 2i)^6$
89. $(5 + \sqrt{-16})^3$
90. $(5 + \sqrt{-9})^3$
91. $(4 + \sqrt{3}i)^4$
92. $(5 - \sqrt{3}i)^4$
93. $\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^3$
94. $\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^3$
95. $\left(\dfrac{1}{4} - \dfrac{\sqrt{3}}{4}i\right)^3$
96. $\left(\dfrac{1}{3} - \dfrac{\sqrt{3}}{3}i\right)^3$

Approximation **In Exercises 97–100, use the Binomial Theorem to approximate the quantity accurate to three decimal places. For example, in Exercise 97, use the expansion**

$$(1.02)^8 = (1 + 0.02)^8 = 1 + 8(0.02) + 28(0.02)^2 + \cdots.$$

97. $(1.02)^8$
98. $(2.005)^{10}$
99. $(2.99)^{12}$
100. $(1.98)^9$

Graphical Reasoning **In Exercises 101 and 102, use a graphing utility to graph f and g in the same viewing window. What is the relationship between the two graphs? Use the Binomial Theorem to write the polynomial function g in standard form.**

101. $f(x) = x^3 - 4x, \quad g(x) = f(x + 3)$
102. $f(x) = -x^4 + 4x^2 - 1, \quad g(x) = f(x - 5)$

Graphical Reasoning **In Exercises 103 and 104, use a graphing utility to graph the functions in the given order and in the same viewing window. Compare the graphs. Which two functions have identical graphs, and why?**

103. (a) $f(x) = (1 - x)^3$
(b) $g(x) = 1 - 3x$
(c) $h(x) = 1 - 3x + 3x^2$
(d) $p(x) = 1 - 3x + 3x^2 - x^3$

104. (a) $f(x) = \left(1 - \frac{1}{2}x\right)^4$
(b) $g(x) = 1 - 2x + \frac{3}{2}x^2$
(c) $h(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3$
(d) $p(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

Probability **In Exercises 105–108, consider n independent trials of an experiment in which each trial has two possible outcomes, success or failure. The probability of a success on each trial is p and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k\, p^k q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of k successes in the n trials of the experiment.**

105. A fair coin is tossed seven times. To find the probability of obtaining four heads, evaluate the term

$${}_7C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^3$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^7$.

106. The probability of a baseball player getting a hit during any given time at bat is $\frac{1}{4}$. To find the probability that the player gets three hits during the next 10 times at bat, evaluate the term

$${}_{10}C_3\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^7$$

in the expansion of $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$.

107. The probability of a sales representative making a sale with any one customer is $\frac{1}{3}$. The sales representative makes eight contacts a day. To find the probability of making four sales, evaluate the term

$${}_8C_4\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^4$$

in the expansion of $\left(\frac{1}{3} + \frac{2}{3}\right)^8$.

108. To find the probability that the sales representative in Exercise 107 makes four sales if the probability of a sale with any one customer is $\frac{1}{2}$, evaluate the term

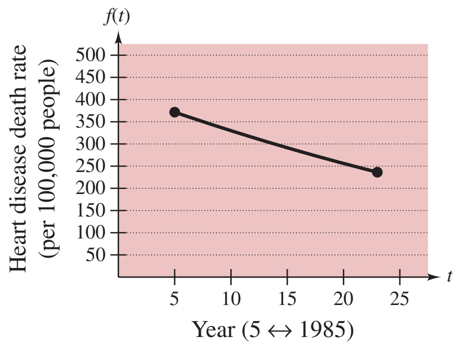
$${}_8C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^4$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

109. ***Health*** The death rates f (per 100,000 people) attributed to heart disease in the United States from 1985 to 2003 can be modeled by the equation

$$f(t) = 0.064t^2 - 9.30t + 416.5, \quad 5 \le t \le 23$$

where t represents the year, with $t = 5$ corresponding to 1985 (see figure). (Source: U.S. National Center for Health Statistics)



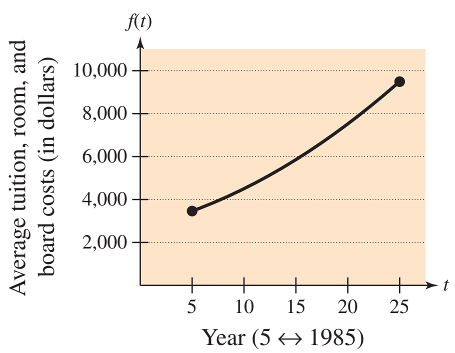
(a) Adjust the model so that $t = 0$ corresponds to 2000 rather than $t = 5$ corresponding to 1985. To do this, shift the graph of f 20 units to the left and obtain $g(t) = f(t + 20)$. Write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.

110. ***Education*** The average tuition, room, and board costs f (in dollars) for undergraduates at public institutions from 1985 through 2005 can be approximated by the model

$$f(t) = 6.22t^2 + 115.2t + 2730, \quad 5 \le t \le 25$$

where t represents the year, with $t = 5$ corresponding to 1985 (see figure). (Source: National Center for Education Statistics)



(a) Adjust the model so that $t = 0$ corresponds to 2000 rather than $t = 5$ corresponding to 1985. To do this, shift the graph of f 20 units to the left and obtain $g(t) = f(t + 20)$. Write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.

Synthesis

True or False? **In Exercises 111 and 112, determine whether the statement is true or false. Justify your answer.**

111. One of the terms in the expansion of $(x - 2y)^{12}$ is $7920x^4y^8$.

112. The x^{10}-term and the x^{14}-term in the expansion of $(x^2 + 3)^{12}$ have identical coefficients.

113. ***Writing*** In your own words, explain how to form the rows of Pascal's Triangle.

114. Form rows 8–10 of Pascal's Triangle.

115. ***Think About It*** How do the expansions of $(x + y)^n$ and $(x - y)^n$ differ?

116. ***Error Analysis*** You are a math instructor and receive the following solutions from one of your students on a quiz. Find the error(s) in each solution and write a short paragraph discussing ways that your student could avoid the error(s) in the future.

(a) Find the second term in the expansion of $(2x - 3y)^5$.

$\cancel{5(2x)^4(3y)^2 = 720x^4y^2}$

(b) Find the fourth term in the expansion of $\left(\frac{1}{2}x + 7y\right)^6$.

$\cancel{{}_6C_4\left(\tfrac{1}{2}x\right)^2(7y)^4 = 9003.75x^2y^4}$

Proof **In Exercises 117–120, prove the property for all integers r and n, where $0 \le r \le n$.**

117. ${}_nC_r = {}_nC_{n-r}$

118. ${}_nC_0 - {}_nC_1 + {}_nC_2 - \cdots \pm {}_nC_n = 0$

119. ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

120. The sum of the numbers in the nth row of Pascal's Triangle is 2^n.

Skills Review

In Exercises 121–124, describe the relationship between the graphs of f and g.

121. $g(x) = f(x) + 8$

122. $g(x) = f(x - 3)$

123. $g(x) = f(-x)$

124. $g(x) = -f(x)$

In Exercises 125 and 126, find the inverse of the matrix.

125. $\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

126. $\begin{bmatrix} 1.2 & -2.3 \\ -2 & 4 \end{bmatrix}$

8.6 Counting Principles

What you should learn

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve more complicated counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.

Why you should learn it

You can use counting principles to solve counting problems that occur in real life. For instance, in Exercise 62 on page 636, you are asked to use counting principles to determine in how many ways a player can select six numbers in a Powerball lottery.

William Thomas Cain/Getty Images

Simple Counting Problems

The last two sections of this chapter present a brief introduction to some of the basic counting principles and their application to probability. In the next section, you will see that much of probability has to do with counting the number of ways an event can occur.

Example 1 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box, its number is written down, and the piece of paper is *returned to the box*. Then, a second piece of paper is drawn from the box, and its number is written down. Finally, the two numbers are added together. In how many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the number of different ways that a sum of 12 can be obtained using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

From this list, you can see that a sum of 12 can occur in five different ways.

CHECKPOINT Now try Exercise 7.

Example 2 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. Two pieces of paper are drawn from the box *at the same time*, and the numbers on the pieces of paper are written down and totaled. In how many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the number of different ways that a sum of 12 can be obtained using two *different* numbers from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can be obtained in four different ways.

CHECKPOINT Now try Exercise 8.

STUDY TIP

The difference between the counting problems in Examples 1 and 2 can be described by saying that the random selection in Example 1 occurs **with replacement,** whereas the random selection in Example 2 occurs **without replacement,** which eliminates the possibility of choosing two 6's.

The Fundamental Counting Principle

Examples 1 and 2 describe simple counting problems in which you can *list* each possible way that an event can occur. When it is possible, this is always the best way to solve a counting problem. However, some events can occur in so many different ways that it is not feasible to write out the entire list. In such cases, you must rely on formulas and counting principles. The most important of these is the **Fundamental Counting Principle.**

> **Fundamental Counting Principle**
>
> Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

Example 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

Solution

There are two events in this situation. The first event is the choice of the first letter, and the second event is the choice of the second letter. Because the English alphabet contains 26 letters, it follows that the number of two-letter pairs is

$$26 \cdot 26 = 676.$$

CHECKPOINT Now try Exercise 9.

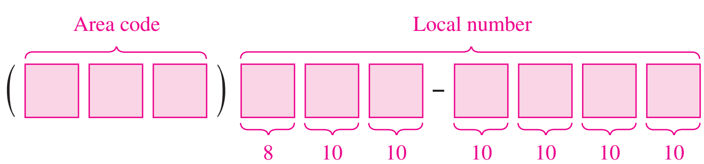
> **STUDY TIP**
>
> The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1, E_2, and E_3 can occur is $m_1 \cdot m_2 \cdot m_3$.

Example 4 Using the Fundamental Counting Principle

Telephone numbers in the United States currently have 10 digits. The first three are the *area code* and the next seven are the *local telephone number*. How many different telephone numbers are possible within each area code? (Note that at this time, a local telephone number cannot begin with 0 or 1.)

Solution

Because the first digit cannot be 0 or 1, there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of local telephone numbers that are possible within each area code is $8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000$.

CHECKPOINT Now try Exercise 15.

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

> **Definition of Permutation**
>
> A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example 5 Finding the Number of Permutations of n Elements

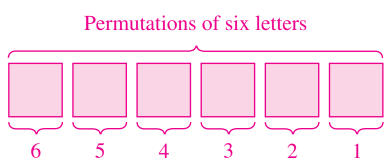
How many permutations are possible of the letters A, B, C, D, E, and F?

Solution

Consider the following reasoning.

First position:	Any of the *six* letters
Second position:	Any of the remaining *five* letters
Third position:	Any of the remaining *four* letters
Fourth position:	Any of the remaining *three* letters
Fifth position:	Any of the remaining *two* letters
Sixth position:	The *one* remaining letter

So, the numbers of choices for the six positions are as follows.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

CHECKPOINT Now try Exercise 33.

> **Number of Permutations of n Elements**
>
> The number of permutations of n elements is given by
>
> $$n \cdot (n - 1) \cdot \cdot \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$
>
> In other words, there are $n!$ different ways that n elements can be ordered.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you might want to choose and order r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time.**

Example 6 Counting Horse Race Finishes

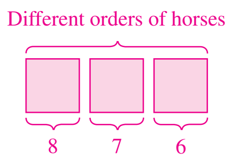
Eight horses are running in a race. In how many different ways can these horses come in first, second, and third? (Assume that there are no ties.)

Solution

Here are the different possibilities.

Win (first position):	*Eight* choices
Place (second position):	*Seven* choices
Show (third position):	*Six* choices

The numbers of choices for the three positions are as follows.



So, using the Fundamental Counting Principle, you can determine that there are

$$8 \cdot 7 \cdot 6 = 336$$

different ways in which the eight horses can come in first, second, and third.

CHECKPOINT Now try Exercise 37.

Permutations of n Elements Taken r at a Time

The number of **permutations of n elements taken r at a time** is given by

$$\begin{aligned} {}_nP_r &= \frac{n!}{(n-r)!} \\ &= n(n-1)(n-2) \cdot \cdot \cdot (n-r+1). \end{aligned}$$

Using this formula, you can rework Example 6 to find that the number of permutations of eight horses taken three at a time is

$$\begin{aligned} {}_8P_3 &= \frac{8!}{5!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} \\ &= 336 \end{aligned}$$

which is the same answer obtained in the example.

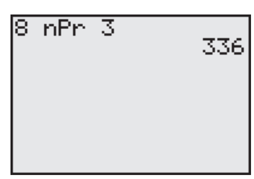


Figure 8.16

TECHNOLOGY TIP Most graphing utilities are programmed to evaluate ${}_nP_r$. Figure 8.16 shows how one graphing utility evaluates the permutation ${}_8P_3$. For instructions on how to use the ${}_nP_r$ feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Remember that for permutations, order is important. So, if you are looking at the possible permutations of the letters A, B, C, and D taken three at a time, the permutations (A, B, D) and (B, A, D) would be counted as different because the *order* of the elements is different.

Suppose, however, that you are asked to find the possible permutations of the letters A, A, B, and C. The total number of permutations of the four letters would be ${}_4P_4 = 4!$. However, not all of these arrangements would be *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdot \cdot \cdot + n_k.$$

The number of **distinguishable permutations** of the n objects is given by

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_k!}.$$

Example 7 Distinguishable Permutations

In how many distinguishable ways can the letters in BANANA be written?

Solution

This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways in which the letters can be written is

$$\frac{6!}{3! \cdot 2! \cdot 1!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!} = 60.$$

The 60 different distinguishable permutations are as follows.

AAABNN	AAANBN	AAANNB	AABANN
AABNAN	AABNNA	AANABN	AANANB
AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN
ABNANA	ABNNAA	ANAABN	ANAANB
ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB
ANNABA	ANNBAA	BAAANN	BAANAN
BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA
NAAABN	NAAANB	NAABAN	NAABNA
NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA
NBAAAN	NBAANA	NBANAA	NBNAAA
NNAAAB	NNAABA	NNABAA	NNBAAA

CHECKPOINT Now try Exercise 45.

Combinations

When you count the number of possible permutations of a set of elements, order is important. As a final topic in this section, you will look at a method for selecting subsets of a larger set in which order *is not* important. Such subsets are called **combinations of *n* elements taken *r* at a time.** For instance, the combinations

$\{A, B, C\}$ and $\{B, A, C\}$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important. So, you would count only one of the two sets. A common example of a combination is a card game in which the player is free to reorder the cards after they have been dealt.

Example 8 Combinations of *n* Elements Taken *r* at a Time

In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)

Solution

The following subsets represent the different combinations of three letters that can be chosen from five letters.

$\{A, B, C\}$ $\{A, B, D\}$

$\{A, B, E\}$ $\{A, C, D\}$

$\{A, C, E\}$ $\{A, D, E\}$

$\{B, C, D\}$ $\{B, C, E\}$

$\{B, D, E\}$ $\{C, D, E\}$

From this list, you can conclude that there are 10 different ways in which three letters can be chosen from five letters.

CHECKPOINT Now try Exercise 57.

Combinations of *n* Elements Taken *r* at a Time

The number of **combinations of *n* elements taken *r* at a time** is given by

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

Note that the formula for ${}_nC_r$ is the same one given for binomial coefficients. To see how this formula is used, solve the counting problem in Example 8. In that problem, you are asked to find the number of combinations of five elements taken three at a time. So, $n = 5$, $r = 3$, and the number of combinations is

$${}_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{2} \cdot 1 \cdot \cancel{3!}} = 10$$

which is the same answer obtained in Example 8.

STUDY TIP

When solving problems involving counting principles, you need to be able to distinguish among the various counting principles in order to determine which is necessary to solve the problem correctly. To do this, ask yourself the following questions.

1. Is the order of the elements important? *Permutation*
2. Are the chosen elements a subset of a larger set in which order is not important? *Combination*
3. Does the problem involve two or more separate events? *Fundamental Counting Principle*

Example 9 Counting Card Hands

A standard poker hand consists of five cards dealt from a deck of 52. How many different poker hands are possible? (After the cards are dealt, the player may reorder them, so order is not important.)

Solution

You can find the number of different poker hands by using the formula for the number of combinations of 52 elements taken five at a time, as follows.

$$\begin{aligned}
{}_{52}C_5 &= \frac{52!}{47!5!} \\
&= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} \\
&= 2{,}598{,}960
\end{aligned}$$

CHECKPOINT Now try Exercise 59.

Example 10 Forming a Team

You are forming a 12-member swim team from 10 girls and 15 boys. The team must consist of five girls and seven boys. How many different 12-member teams are possible?

Solution

There are ${}_{10}C_5$ ways of choosing five girls. There are ${}_{15}C_7$ ways of choosing seven boys. By the Fundamental Counting Principle, there are ${}_{10}C_5 \cdot {}_{15}C_7$ ways of choosing five girls and seven boys.

$$\begin{aligned}
{}_{10}C_5 \cdot {}_{15}C_7 &= \frac{10!}{5! \cdot 5!} \cdot \frac{15!}{8! \cdot 7!} \\
&= 252 \cdot 6435 \\
&= 1{,}621{,}620
\end{aligned}$$

So, there are 1,621,620 12-member swim teams possible. You can verify this by using the ${}_nC_r$ feature of a graphing utility, as shown in Figure 8.17.

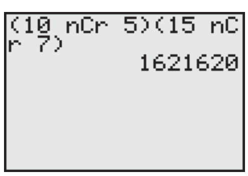


Figure 8.17

CHECKPOINT Now try Exercise 69.

8.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The _______ states that if there are m_1 ways for one event to occur and m_2 ways for a second event to occur, then there are $m_1 \cdot m_2$ ways for both events to occur.
2. An ordering of n elements is called a _______ of the elements.
3. The number of permutations of n elements taken r at a time is given by the formula _______ .
4. The number of _______ of n objects is given by $\dfrac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdot \cdot \cdot \cdot n_k!}$.
5. When selecting subsets of a larger set in which order is not important, you are finding the number of _______ of n elements taken r at a time.

Random Selection **In Exercises 1–8, determine the number of ways in which a computer can randomly generate one or more such integers, or pairs of integers, from 1 through 12.**

1. An odd integer
2. An even integer
3. A prime integer
4. An integer that is greater than 6
5. An integer that is divisible by 4
6. An integer that is divisible by 3
7. A pair of integers whose sum is 8
8. A pair of distinct integers whose sum is 8

9. ***Consumer Awareness*** A customer can choose one of four amplifiers, one of six compact disc players, and one of five speaker models for an entertainment system. Determine the number of possible system configurations.
10. ***Course Schedule*** A college student is preparing a course schedule for the next semester. The student must select one of two mathematics courses, one of three science courses, and one of five courses from the social sciences and humanities. How many schedules are possible?
11. ***True-False Exam*** In how many ways can a 10-question true-false exam be answered? (Assume that no questions are omitted.)
12. ***Attaché Case*** An attaché case has two locks, each of which is a three-digit number sequence where digits may be repeated. Find the total number of combinations of the two locks in order to open the attaché case.
13. ***Three-Digit Numbers*** How many three-digit numbers can be formed under each condition?
    (a) The leading digit cannot be a 0.
    (b) The leading digit cannot be a 0 and no repetition of digits is allowed.
14. ***Four-Digit Numbers*** How many four-digit numbers can be formed under each condition?
    (a) The leading digit cannot be a 0 and the number must be less than 5000.
    (b) The leading digit cannot be a 0 and the number must be even.
15. ***Telephone Numbers*** In 2006, the state of Nevada had two area codes. Using the information about telephone numbers given in Example 4, how many telephone numbers could Nevada's phone system have accommodated?
16. ***Telephone Numbers*** In 2006, the state of Kansas had four area codes. Using the information about telephone numbers given in Example 4, how many telephone numbers could Kansas's phone system have accommodated?
17. ***Radio Stations*** Typically radio stations are identified by four "call letters." Radio stations east of the Mississippi River have call letters that start with the letter W and radio stations west of the Mississippi River have call letters that start with the letter K.
    (a) Find the number of different sets of radio station call letters that are possible in the United States.
    (b) Find the number of different sets of radio station call letters that are possible if the call letters must include a Q.
18. ***PIN Codes*** ATM personal identification number (PIN) codes typically consist of four-digit sequences of numbers.
    (a) Find the total number of ATM codes possible.
    (b) Find the total number of ATM codes possible if the first digit is not a 0.
19. ***ZIP Codes*** In 1963, the United States Postal Service launched the Zoning Improvement Plan (ZIP) Code to streamline the mail-delivery system. A ZIP code consists of a five-digit sequence of numbers.

(a) Find the number of ZIP codes consisting of five digits.

(b) Find the number of ZIP codes consisting of five digits, if the first digit is a 1 or a 2.

20. ***ZIP Codes*** In 1983, in order to identify small geographic segments, such as city blocks or a single building, within a delivery code, the post office began to use an expanded ZIP code called ZIP+4, which is composed of the original five-digit code plus a four-digit add-on code.

(a) Find the number of ZIP codes consisting of five digits followed by the four additional digits.

(b) Find the number of ZIP codes consisting of five digits followed by the four additional digits, if the first number of the five-digit code is a 1 or a 2.

21. ***Entertainment*** Three couples have reserved seats in a row for a concert. In how many different ways can they be seated if

(a) there are no seating restrictions?

(b) the two members of each couple wish to sit together?

22. ***Single File*** In how many orders can five girls and three boys walk through a doorway single file if

(a) there are no restrictions?

(b) the girls walk through before the boys?

In Exercises 23–28, evaluate $_nP_r$ using the formula from this section.

23. $_4P_4$ **24.** $_5P_5$

25. $_8P_3$ **26.** $_{20}P_2$

27. $_5P_4$ **28.** $_7P_4$

In Exercises 29–32, evaluate $_nP_r$ using a graphing utility.

29. $_{20}P_6$ **30.** $_{10}P_8$

31. $_{120}P_4$ **32.** $_{100}P_5$

33. ***Posing for a Photograph*** In how many ways can five children posing for a photograph line up in a row?

34. ***Riding in a Car*** In how many ways can four people sit in a four-passenger car?

35. ***U.S. Supreme Court*** The nine justices of the U.S. Supreme Court pose for a photograph while standing in a straight line, as opposed to the typical pose of two rows. How many different orders of the justices are possible for this photograph?

36. ***Manufacturing*** Four processes are involved in assembling a product, and they can be performed in any order. The management wants to test each order to determine which is the least time-consuming. How many different orders will have to be tested?

37. C***hoosing Officers*** From a pool of 12 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. In how many ways can the offices be filled?

38. ***Batting Order*** How many different batting orders can a baseball coach create from a team of 15 players, if there are nine positions to fill?

39. ***School Locker*** A school locker has a dial lock on which there are 37 numbers from 0 to 36. Find the total number of possible lock combinations, if the lock requires a three-digit sequence of left-right-left and the numbers can be repeated.

40. ***Sports*** Eight sprinters have qualified for the finals in the 100-meter dash at the NCAA national track meet. How many different orders of the top three finishes are possible? (Assume there are no ties.)

In Exercises 41 and 42, use the letters A, B, C, and D.

41. Write all permutations of the letters.

42. Write all permutations of the letters if the letters B and C must remain between the letters A and D.

In Exercises 43–46, find the number of distinguishable permutations of the group of letters.

43. A, A, G, E, E, E, M **44.** B, B, B, T, T, T, T, T

45. A, L, G, E, B, R, A

46. M, I, S, S, I, S, S, I, P, P, I

In Exercises 47–52, evaluate $_nC_r$ using the formula from this section.

47. $_5C_2$ **48.** $_6C_3$

49. $_4C_1$ **50.** $_5C_1$

51. $_{25}C_0$ **52.** $_{20}C_0$

In Exercises 53–56, evaluate $_nC_r$ using a graphing utility.

53. $_{20}C_4$ **54.** $_{10}C_7$

55. $_{42}C_5$ **56.** $_{50}C_6$

In Exercises 57 and 58, use the letters A, B, C, D, E, and F.

57. Write all possible selections of two letters that can be formed from the letters. (The order of the two letters is not important.)

58. Write all possible selections of three letters that can be formed from the letters. (The order of the three letters is not important.)

59. ***Forming a Committee*** As of June 2006, the U.S. Senate Committee on Indian Affairs had 14 members. If party affiliation is not a factor in selection, how many different committees are possible from the 100 U.S. senators?

60. ***Exam Questions*** You can answer any 12 questions from a total of 14 questions on an exam. In how many different ways can you select the questions?

61. ***Lottery*** In Washington's Lotto game, a player chooses six distinct numbers from 1 to 49. In how many ways can a player select the six numbers?

62. ***Lottery*** Powerball is played with 55 white balls, numbered 1 through 55, and 42 red balls, numbered 1 through 42. Five white balls and one red ball, the Powerball, are drawn. In how many ways can a player select the six numbers?

63. ***Geometry*** Three points that are not collinear determine three lines. How many lines are determined by nine points, no three of which are collinear?

64. ***Defective Units*** A shipment of 25 television sets contains three defective units. In how many ways can a vending company purchase four of these units and receive (a) all good units, (b) two good units, and (c) at least two good units?

65. ***Poker Hand*** You are dealt five cards from an ordinary deck of 52 playing cards. In how many ways can you get a full house? (A full house consists of three of one kind and two of another. For example, 8-8-8-5-5 and K-K-K-10-10 are full houses.)

66. ***Card Hand*** Five cards are chosen from a standard deck of 52 cards. How many five-card combinations contain two jacks and three aces?

67. ***Job Applicants*** A clothing manufacturer interviews 12 people for four openings in the human resources department of the company. Five of the 12 people are women. If all 12 are qualified, in how many ways can the employer fill the four positions if (a) the selection is random and (b) exactly two women are selected?

68. ***Job Applicants*** A law office interviews paralegals for 10 openings. There are 13 paralegals with two years of experience and 20 paralegals with one year of experience. How many combinations of seven paralegals with two years of experience and three paralegals with one year of experience are possible?

69. ***Forming a Committee*** A six-member research committee is to be formed having one administrator, three faculty members, and two students. There are seven administrators, 12 faculty members, and 20 students in contention for the committee. How many six-member committees are possible?

70. ***Interpersonal Relationships*** The number of possible interpersonal relationships increases dramatically as the size of a group increases. Determine the number of different two-person relationships that are possible in a group of people of size (a) 3, (b) 8, (c) 12, and (d) 20.

Geometry **In Exercises 71–74, find the number of diagonals of the polygon. (A line segment connecting any two nonadjacent vertices is called a *diagonal* of a polygon.)**

71. Pentagon **72.** Hexagon

73. Octagon **74.** Decagon

In Exercises 75–82, solve for *n*.

75. $14 \cdot {}_nP_3 = {}_{n+2}P_4$

76. ${}_nP_5 = 18 \cdot {}_{n-2}P_4$

77. ${}_nP_4 = 10 \cdot {}_{n-1}P_3$

78. ${}_nP_6 = 12 \cdot {}_{n-1}P_5$

79. ${}_{n+1}P_3 = 4 \cdot {}_nP_2$

80. ${}_{n+2}P_3 = 6 \cdot {}_{n+2}P_1$

81. $4 \cdot {}_{n+1}P_2 = {}_{n+2}P_3$

82. $5 \cdot {}_{n-1}P_1 = {}_nP_2$

Synthesis

True or False? **In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.**

83. The number of pairs of letters that can be formed from any of the first 13 letters in the alphabet (A–M), where repetitions are allowed, is an example of a permutation.

84. The number of permutations of n elements can be derived by using the Fundamental Counting Principle.

85. ***Think About It*** Can your calculator evaluate ${}_{100}P_{80}$? If not, explain why.

86. ***Writing*** Explain in your own words the meaning of ${}_nP_r$.

87. What is the relationship between ${}_nC_r$ and ${}_nC_{n-r}$?

88. Without calculating the numbers, determine which of the following is greater. Explain.

(a) The number of combinations of 10 elements taken six at a time

(b) The number of permutations of 10 elements taken six at a time

Proof **In Exercises 89–92, prove the identity.**

89. ${}_nP_{n-1} = {}_nP_n$

90. ${}_nC_n = {}_nC_0$

91. ${}_nC_{n-1} = {}_nC_1$

92. ${}_nC_r = \dfrac{{}_nP_r}{r!}$

Skills Review

In Exercises 93–96, solve the equation. Round your answer to three decimal places, if necessary.

93. $\sqrt{x-3} = x - 6$

94. $\dfrac{4}{t} + \dfrac{3}{2t} = 1$

95. $\log_2(x-3) = 5$

96. $e^{x/3} = 16$

In Exercises 97–100, use Cramer's Rule to solve the system of equations.

97. $\begin{cases} -5x + 3y = -14 \\ 7x - 2y = 2 \end{cases}$

98. $\begin{cases} 8x + y = 35 \\ 6x + 2y = 10 \end{cases}$

99. $\begin{cases} -3x - 4y = -1 \\ 9x + 5y = -4 \end{cases}$

100. $\begin{cases} 10x - 11y = -74 \\ -8x - 4y = 8 \end{cases}$

8.7 Probability

The Probability of an Event

Any happening whose result is uncertain is called an **experiment.** The possible results of the experiment are **outcomes,** the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event.**

For instance, when a six-sided die is tossed, the sample space can be represented by the numbers 1 through 6. For the experiment to be fair, each of the outcomes is *equally likely*.

To describe a sample space in such a way that each outcome is equally likely, you must sometimes distinguish between or among various outcomes in ways that appear artificial. Example 1 illustrates such a situation.

What you should learn

- Find probabilities of events.
- Find probabilities of mutually exclusive events.
- Find probabilities of independent events.
- Find probabilities of complements of events.

Why you should learn it

You can use probability to solve a variety of problems that occur in real life. For instance, in Exercise 31 on page 646, you are asked to use probability to help analyze the age distribution of unemployed workers.

Tony Freeman/PhotoEdit

Example 1 Finding the Sample Space

Find the sample space for each of the following.

a. One coin is tossed.

b. Two coins are tossed.

c. Three coins are tossed.

Solution

a. Because the coin will land either heads up (denoted by H) or tails up (denoted by T), the sample space S is

$$S = \{H, T\}.$$

b. Because either coin can land heads up or tails up, the possible outcomes are as follows.

$HH =$ heads up on both coins

$HT =$ heads up on first coin and tails up on second coin

$TH =$ tails up on first coin and heads up on second coin

$TT =$ tails up on both coins

So, the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Note that this list distinguishes between the two cases HT and TH, even though these two outcomes appear to be similar.

c. Following the notation of part (b), the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Note that this list distinguishes among the cases HHT, HTH, and THH, and among the cases HTT, THT, and TTH.

CHECKPOINT Now try Exercise 1.

To calculate the probability of an event, count the number of outcomes in the event and in the sample space. The *number of equally likely outcomes* in event E is denoted by $n(E)$, and the number of equally likely outcomes in the sample space S is denoted by $n(S)$. The probability that event E will occur is given by $n(E)/n(S)$.

Exploration

Toss two coins 40 times and write down the number of heads that occur on each toss (0, 1, or 2). How many times did two heads occur? How many times would you expect two heads to occur if you did the experiment 1000 times?

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, the **probability** of event E is given by

$$P(E) = \frac{n(E)}{n(S)}.$$

Because the number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, the probability of an event must be a number from 0 to 1, inclusive. That is,

$$0 \leq P(E) \leq 1$$

as indicated in Figure 8.18. If $P(E) = 0$, event E *cannot occur*, and E is called an **impossible event.** If $P(E) = 1$, event E *must occur*, and E is called a **certain event.**

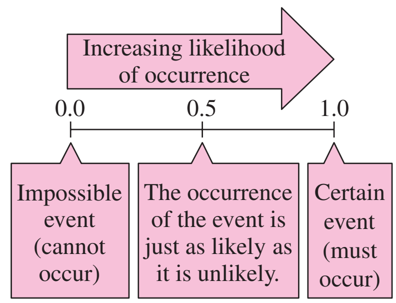


Figure 8.18

Example 2 Finding the Probability of an Event

a. Two coins are tossed. What is the probability that both land heads up?

b. A card is drawn from a standard deck of playing cards. What is the probability that it is an ace?

Solution

a. Following the procedure in Example 1(b), let

$$E = \{HH\}$$

and

$$S = \{HH, HT, TH, TT\}.$$

The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

b. Because there are 52 cards in a standard deck of playing cards and there are four aces (one of each suit), the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{4}{52} = \frac{1}{13}.$$

CHECKPOINT Now try Exercise 7.

STUDY TIP

You can write a probability as a fraction, a decimal, or a percent. For instance, in Example 2(a), the probability of getting two heads can be written as $\frac{1}{4}$, 0.25, or 25%.

Example 3 Finding the Probability of an Event

Two six-sided dice are tossed. What is the probability that a total of 7 is rolled? (See Figure 8.19.)

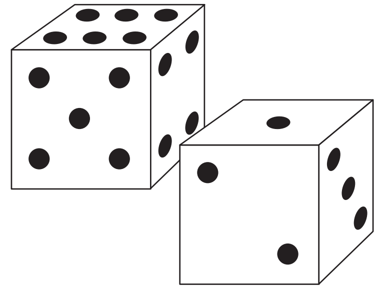


Figure 8.19

Solution

Because there are six possible outcomes on each die, you can use the Fundamental Counting Principle to conclude that there are $6 \cdot 6 = 36$ different outcomes when two dice are tossed. To find the probability of rolling a total of 7, you must first count the number of ways this can occur.

First die	1	2	3	4	5	6
Second die	6	5	4	3	2	1

So, a total of 7 can be rolled in six ways, which means that the probability of rolling a 7 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

CHECKPOINT Now try Exercise 15.

You could have written out each sample space in Examples 2 and 3 and simply counted the outcomes in the desired events. For larger sample spaces, however, using the counting principles discussed in Section 8.6 should save you time.

Example 4 Finding the Probability of an Event

Twelve-sided dice, as shown in Figure 8.20, can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 appears twice on each die. Show that these dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of different outcomes.

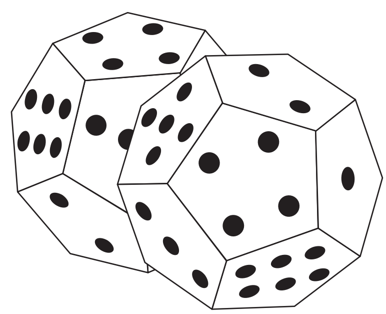


Figure 8.20

Solution

For an ordinary six-sided die, each of the numbers 1, 2, 3, 4, 5, and 6 occurs only once, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For a 12-sided die, each number occurs twice, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

CHECKPOINT Now try Exercise 17.

Example 5 The Probability of Winning a Lottery

In Delaware's Multi-Win Lotto game, a player chooses six different numbers from 1 to 35. If these six numbers match the six numbers drawn (in any order) by the lottery commission, the player wins (or shares) the top prize. What is the probability of winning the top prize if the player buys one ticket?

Solution

To find the number of elements in the sample space, use the formula for the number of combinations of 35 elements taken six at a time.

$$n(S) = {}_{35}C_6$$
$$= \frac{35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 \cdot 30}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1{,}623{,}160$$

If a person buys only one ticket, the probability of winning is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{1{,}623{,}160}.$$

CHECKPOINT Now try Exercise 19.

Prerequisite Skills

Review combinations of n elements taken r at a time in Section 8.6, if you have difficulty with this example.

Example 6 Random Selection

The numbers of colleges and universities in various regions of the United States in 2004 are shown in Figure 8.21. One institution is selected at random. What is the probability that the institution is in one of the three southern regions? (Source: U.S. National Center for Education Statistics)

Solution

From the figure, the total number of colleges and universities is 4231. Because there are $398 + 287 + 720 = 1405$ colleges and universities in the three southern regions, the probability that the institution is in one of these regions is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1405}{4231} \approx 0.33.$$

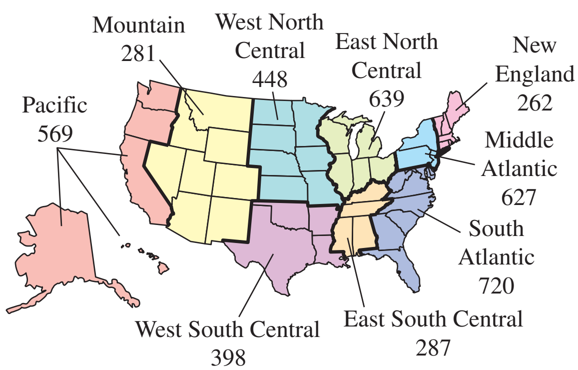


Figure 8.21

CHECKPOINT Now try Exercise 31.

Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** if A and B have no outcomes in common. In the terminology of sets, the intersection of A and B is the empty set, which is expressed as

$$P(A \cap B) = 0.$$

For instance, if two dice are tossed, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive. To find the probability that one or the other of two mutually exclusive events will occur, you can *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, the probability of A *or* B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Example 7 The Probability of a Union

One card is selected from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

Solution

Because the deck has 13 hearts, the probability of selecting a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, because the deck has 12 face cards, the probability of selecting a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Because three of the cards are hearts and face cards (see Figure 8.22), it follows that

$$P(A \cap B) = \frac{3}{52}.$$

Finally, applying the formula for the probability of the union of two events, you can conclude that the probability of selecting a heart or a face card is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} \approx 0.42.$$

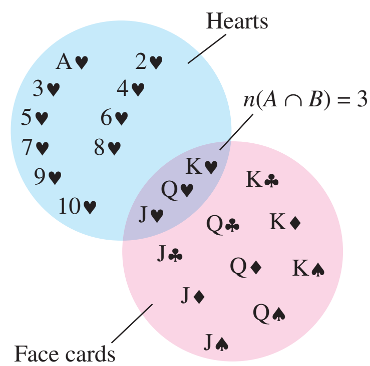


Figure 8.22

CHECKPOINT Now try Exercise 49.

Example 8 Probability of Mutually Exclusive Events

The personnel department of a company has compiled data on the numbers of employees who have been with the company for various periods of time. The results are shown in the table.

Years of service	Numbers of employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	37
35–39	21
40–44	8

If an employee is chosen at random, what is the probability that the employee has (a) 4 or fewer years of service and (b) 9 or fewer years of service?

Solution

a. To begin, add the number of employees and find that the total is 529. Next, let event A represent choosing an employee with 0 to 4 years of service. Then the probability of choosing an employee who has 4 or fewer years of service is

$$P(A) = \frac{157}{529} \approx 0.30.$$

b. Let event B represent choosing an employee with 5 to 9 years of service. Then

$$P(B) = \frac{89}{529}.$$

Because event A from part (a) and event B have no outcomes in common, you can conclude that these two events are mutually exclusive and that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{157}{529} + \frac{89}{529} \\ &= \frac{246}{529} \\ &\approx 0.47. \end{aligned}$$

So, the probability of choosing an employee who has 9 or fewer years of service is about 0.47.

CHECKPOINT Now try Exercise 51.

Independent Events

Two events are **independent** if the occurrence of one has no effect on the occurrence of the other. For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. To find the probability that two independent events will occur, *multiply* the probabilities of each.

> **Probability of Independent Events**
>
> If A and B are **independent events,** the probability that both A and B will occur is given by
>
> $$P(A \text{ and } B) = P(A) \cdot P(B).$$

Example 9 Probability of Independent Events

A random number generator on a computer selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?

Solution

The probability of selecting a number from 1 to 5 is

$$P(A) = \frac{5}{20} = \frac{1}{4}.$$

So, the probability that all three numbers are less than or equal to 5 is

$$P(A) \cdot P(A) \cdot P(A) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{64}.$$

CHECKPOINT Now try Exercise 52.

Example 10 Probability of Independent Events

In 2004, approximately 65% of the population of the United States was 25 years old or older. In a survey, 10 people were chosen at random from the population. What is the probability that all 10 were 25 years old or older? (Source: U.S. Census Bureau)

Solution

Let A represent choosing a person who was 25 years old or older. The probability of choosing a person who was 25 years old or older is 0.65, the probability of choosing a second person who was 25 years old or older is 0.65, and so on. Because these events are independent, you can conclude that the probability that all 10 people were 25 years old or older is

$$[P(A)]^{10} = (0.65)^{10} \approx 0.01.$$

CHECKPOINT Now try Exercise 53.

The Complement of an Event

The **complement of an event** A is the collection of all outcomes in the sample space that are *not* in A. The complement of event A is denoted by A'. Because $P(A \text{ or } A') = 1$ and because A and A' are mutually exclusive, it follows that $P(A) + P(A') = 1$. So, the probability of A' is given by

$$P(A') = 1 - P(A).$$

For instance, if the probability of *winning* a game is

$$P(A) = \frac{1}{4}$$

then the probability of *losing* the game is

$$P(A') = 1 - \frac{1}{4}$$

$$= \frac{3}{4}.$$

> **Exploration**
>
> You are in a class with 22 other people. What is the probability that at least two out of the 23 people will have a birthday on the same day of the year? What if you know the probability of everyone having the same birthday? Do you think this information would help you to find the answer?

> **Probability of a Complement**
>
> Let A be an event and let A' be its complement. If the probability of A is $P(A)$, then the probability of the complement is given by
>
> $$P(A') = 1 - P(A).$$

Example 11 Finding the Probability of a Complement

A manufacturer has determined that a machine averages one faulty unit for every 1000 it produces. What is the probability that an order of 200 units will have one or more faulty units?

Solution

To solve this problem as stated, you would need to find the probabilities of having exactly one faulty unit, exactly two faulty units, exactly three faulty units, and so on. However, using complements, you can simply find the probability that all units are perfect and then subtract this value from 1. Because the probability that any given unit is perfect is 999/1000, the probability that all 200 units are perfect is

$$P(A) = \left(\frac{999}{1000}\right)^{200}$$

$$\approx 0.82.$$

So, the probability that at least one unit is faulty is

$$P(A') = 1 - P(A)$$

$$\approx 0.18.$$

CHECKPOINT Now try Exercise 55.

8.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

In Exercises 1–7, fill in the blanks.

1. An _______ is an event whose result is uncertain, and the possible results of the event are called _______ .
2. The set of all possible outcomes of an experiment is called the _______ .
3. To determine the _______ of an event, you can use the formula $P(E) = \dfrac{n(E)}{n(S)}$, where $n(E)$ is the number of outcomes in the event and $n(S)$ is the number of outcomes in the sample space.
4. If $P(E) = 0$, then E is an _______ event, and if $P(E) = 1$, then E is a _______ event.
5. If two events from the same sample space have no outcomes in common, then the two events are _______ .
6. If the occurrence of one event has no effect on the occurrence of a second event, then the events are _______ .
7. The _______ of an event A is the collection of all outcomes in the sample space that are not in A.
8. Match the probability formula with the correct probability name.

(a) Probability of the union of two events	(i) $P(A \cup B) = P(A) + P(B)$
(b) Probability of mutually exclusive events	(ii) $P(A') = 1 - P(A)$
(c) Probability of independent events	(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(d) Probability of a complement	(iv) $P(A \text{ and } B) = P(A) \cdot P(B)$

In Exercises 1–6, determine the sample space for the experiment.

1. A coin and a six-sided die are tossed.
2. A six-sided die is tossed twice and the sum of the results is recorded.
3. A taste tester has to rank three varieties of orange juice, A, B, and C, according to preference.
4. Two marbles are selected (without replacement) from a sack containing two red marbles, two blue marbles, and one yellow marble. The color of each marble is recorded.
5. Two county supervisors are selected from five supervisors, A, B, C, D, and E, to study a recycling plan.
6. A sales representative makes presentations of a product in three homes per day. In each home there may be a sale (denote by S) or there may be no sale (denote by F).

Tossing a Coin **In Exercises 7–10, find the probability for the experiment of tossing a coin three times. Use the sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.**

7. The probability of getting exactly two tails
8. The probability of getting a head on the first toss
9. The probability of getting at least one head
10. The probability of getting at least two heads

Drawing a Card **In Exercises 11–14, find the probability for the experiment of selecting one card from a standard deck of 52 playing cards.**

11. The card is a face card.
12. The card is not a black face card.
13. The card is a face card or an ace.
14. The card is a 9 or lower. (Aces are low.)

Tossing a Die **In Exercises 15–18, find the probability for the experiment of tossing a six-sided die twice.**

15. The sum is 6.
16. The sum is at least 8.
17. The sum is less than 11.
18. The sum is odd or prime.

Drawing Marbles **In Exercises 19–22, find the probability for the experiment of drawing two marbles (without replacement) from a bag containing one green, two yellow, and three red marbles.**

19. Both marbles are red.
20. Both marbles are yellow.
21. Neither marble is yellow.
22. The marbles are of different colors.

In Exercises 23–26, you are given the probability that an event *will* happen. Find the probability that the event *will not* happen.

23. $P(E) = 0.75$

24. $P(E) = 0.\overline{2}$

25. $P(E) = \frac{2}{3}$

26. $P(E) = \frac{7}{8}$

In Exercises 27–30, you are given the probability that an event *will not* happen. Find the probability that the event *will* happen.

27. $P(E') = 0.12$

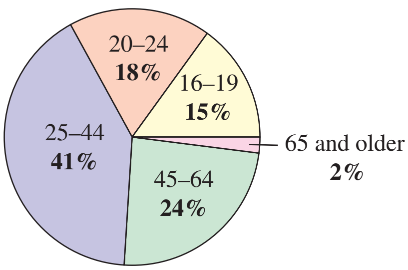
28. $P(E') = 0.84$

29. $P(E') = \frac{13}{20}$

30. $P(E') = \frac{61}{100}$

31. ***Graphical Reasoning*** In 2004, there were approximately 8.15 million unemployed workers in the United States. The circle graph shows the age profile of these unemployed workers. (Source: U.S. Bureau of Labor Statistics)

Ages of Unemployed Workers



(a) Estimate the number of unemployed workers in the 16–19 age group.

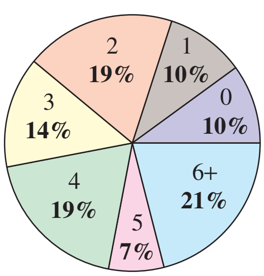
(b) What is the probability that a person selected at random from the population of unemployed workers is in the 25–44 age group?

(c) What is the probability that a person selected at random from the population of unemployed workers is in the 45–64 age group?

(d) What is the probability that a person selected at random from the population of unemployed workers is 45 or older?

32. ***Graphical Reasoning*** The circle graph shows the numbers of children of the 42 U.S. presidents as of 2006. (Source: infoplease.com)

Children of U.S. Presidents



(a) Determine the number of presidents who had no children.

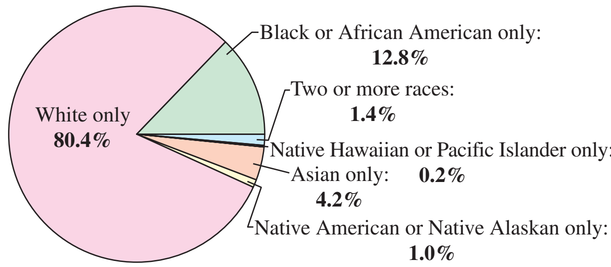
(b) Determine the number of presidents who had four children.

(c) What is the probability that a president selected at random had five or more children?

(d) What is the probability that a president selected at random had three children?

33. ***Graphical Reasoning*** The total population of the United States in 2004 was approximately 293.66 million. The circle graph shows the race profile of the population. (Source: U.S. Census Bureau)

United States Population by Race



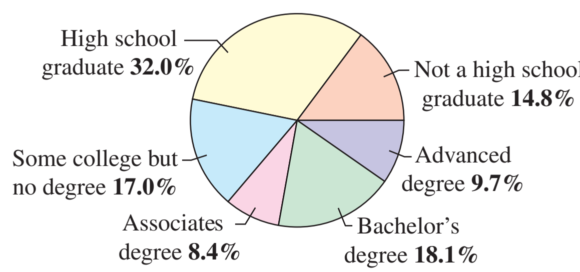
(a) Estimate the population of African Americans.

(b) A person is selected at random. Find the probability that this person is Native American or Native Alaskan.

(c) A person is selected at random. Find the probability that this person is Native American, Native Alaskan, Native Hawaiian, or Pacific Islander.

34. ***Graphical Reasoning*** The educational attainment of the United States population age 25 years or older in 2004 is shown in the circle graph. Use the fact that the population of people 25 years or older was 186.88 million in 2004. (Source: U.S. Census Bureau)

Educational Attainment



(a) Estimate the number of people 25 or older who have high school diplomas.

(b) Estimate the number of people 25 or older who have advanced degrees.

(c) Find the probability that a person 25 or older selected at random has earned a Bachelor's degree or higher.

(d) Find the probability that a person 25 or older selected at random has earned a high school diploma or gone on to post-secondary education.

(e) Find the probability that a person 25 or older selected at random has earned an Associate's degree or higher.

35. ***Data Analysis*** One hundred college students were interviewed to determine their political party affiliations and whether they favored a balanced budget amendment to the Constitution. The results of the study are listed in the table, where D represents Democrat and R represents Republican.

	D	***R***	**Total**
Favor	23	32	55
Oppose	25	9	34
Unsure	7	4	11
Total	55	45	100

A person is selected at random from the sample. Find the probability that the person selected is (a) a person who doesn't favor the amendment, (b) a Republican, (c) a Democrat who favors the amendment.

36. ***Data Analysis*** A study of the effectiveness of a flu vaccine was conducted with a sample of 500 people. Some participants in the study were given no vaccine, some were given one injection, and some were given two injections. The results of the study are shown in the table.

	Flu	**No flu**	**Total**
No vaccine	7	149	156
One injection	2	52	54
Two Injections	13	277	290
Total	22	478	500

A person is selected at random from the sample. Find the probability that the person selected (a) had two injections, (b) did not get the flu, and (c) got the flu and had one injection.

37. ***Alumni Association*** A college sends a survey to selected members of the class of 2006. Of the 1254 people who graduated that year, 672 are women, of whom 124 went on to graduate school. Of the 582 male graduates, 198 went on to graduate school. An alumni member is selected at random. What is the probability that the person is (a) female, (b) male, and (c) female and did not attend graduate school?

38. ***Education*** In a high school graduating class of 128 students, 52 are on the honor roll. Of these, 48 are going on to college; of the other 76 students, 56 are going on to college. A student is selected at random from the class. What is the probability that the person chosen is (a) going to college, (b) not going to college, and (c) not going to college and on the honor roll?

39. ***Election*** Taylor, Moore, and Perez are candidates for public office. It is estimated that Moore and Perez have about the same probability of winning, and Taylor is believed to be twice as likely to win as either of the others. Find the probability of each candidate's winning the election.

40. ***Payroll Error*** The employees of a company work in six departments: 31 are in sales, 54 are in research, 42 are in marketing, 20 are in engineering, 47 are in finance, and 58 are in production. One employee's paycheck is lost. What is the probability that the employee works in the research department?

In Exercises 41–52, the sample spaces are large and you should use the counting principles discussed in Section 6.6.

41. ***Preparing for a Test*** A class is given a list of 20 study problems from which 10 will be chosen as part of an upcoming exam. A given student knows how to solve 15 of the problems. Find the probability that the student will be able to answer (a) all 10 questions on the exam, (b) exactly 8 questions on the exam, and (c) at least 9 questions on the exam.

42. ***Payroll Mix-Up*** Five paychecks and envelopes are addressed to five different people. The paychecks are randomly inserted into the envelopes. What is the probability that (a) exactly one paycheck is inserted in the correct envelope and (b) at least one paycheck is inserted in the correct envelope?

43. ***Game Show*** On a game show you are given five digits to arrange in the proper order to form the price of a car. If you are correct, you win the car. What is the probability of winning if you (a) guess the position of each digit and (b) know the first digit and guess the others?

44. ***Card Game*** The deck of a card game is made up of 108 cards. Twenty-five each are red, yellow, blue, and green, and eight are wild cards. Each player is randomly dealt a seven-card hand. What is the probability that a hand will contain (a) exactly two wild cards, and (b) two wild cards, two red cards, and three blue cards?

45. ***Radio Stations*** Typically radio stations are identified by four "call letters." Radio stations east of the Mississippi River have call letters that start with the letter W and radio stations west of the Mississippi River have call letters that start with the letter K. Assuming the station call letters are equally distributed, what is the probability that a radio station selected at random has call letters that contain (a) a Q and a Y, and (b) a Q, a Y, and an X?

46. ***PIN Codes*** ATM personal identification number (PIN) codes typically consist of four-digit sequences of numbers. Find the probability that if you forget your PIN, you can guess the correct sequence (a) at random and (b) if you recall the first two digits.

47. ***Lottery*** Powerball is played with 55 white balls, numbered 1 through 55, and 42 red balls, numbered 1 through 42. Five white balls and one red ball, the Powerball, are drawn to determine the winning ticket(s). Find the probability that you purchase a winning ticket if you purchase (a) 100 tickets and (b) 1000 tickets with different combinations of numbers.

48. ***ZIP Codes*** The U.S. Postal Service is to deliver a letter to a certain postal ZIP+4 code. Find the probability that the ZIP+4 code is correct if the sender (a) randomly chooses the code, (b) knows the five-digit code but must randomly choose the last four digits, and (c) knows the five-digit code and the first two digits of the plus 4 code.

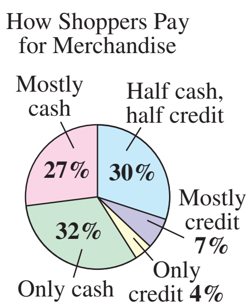
49. ***Drawing a Card*** One card is selected at random from a standard deck of 52 playing cards. Find the probability that (a) the card is an even-numbered card, (b) the card is a heart or a diamond, and (c) the card is a nine or a face card.

50. ***Poker Hand*** Five cards are drawn from an ordinary deck of 52 playing cards. What is the probability of getting a full house? (A full house consists of three of one kind and two of another kind.)

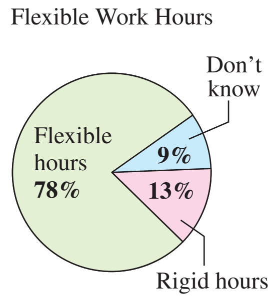
51. ***Defective Units*** A shipment of 12 microwave ovens contains three defective units. A vending company has ordered four of these units, and because all are packaged identically, the selection will be random. What is the probability that (a) all four units are good, (b) exactly two units are good, and (c) at least two units are good?

52. ***Random Number Generator*** Two integers from 1 through 40 are chosen by a random number generator. What is the probability that (a) the numbers are both even, (b) one number is even and one is odd, (c) both numbers are less than 30, and (d) the same number is chosen twice?

53. ***Consumerism*** Suppose that the methods used by shoppers to pay for merchandise are as shown in the circle graph. Two shoppers are chosen at random. What is the probability that both shoppers paid for their purchases only in cash?



54. ***Flexible Work Hours*** In a survey, people were asked if they would prefer to work flexible hours—even if it meant slower career advancement—so they could spend more time with their families. The results of the survey are shown in the circle graph. Three people from the survey are chosen at random. What is the probability that all three people would prefer flexible work hours?



55. ***Backup System*** A space vehicle has an independent backup system for one of its communication networks. The probability that either system will function satisfactorily for the duration of a flight is 0.985. What is the probability that during a given flight (a) both systems function satisfactorily, (b) at least one system functions satisfactorily, and (c) both systems fail?

56. ***Backup Vehicle*** A fire company keeps two rescue vehicles to serve the community. Because of the demand on the vehicles and the chance of mechanical failure, the probability that a specific vehicle is available when needed is 90%. The availability of one vehicle is *independent* of the other. Find the probability that (a) both vehicles are available at a given time, (b) neither vehicle is available at a given time, and (c) at least one vehicle is available at a given time.

57. ***Making a Sale*** A sales representative makes sales on approximately one-fifth of all calls. On a given day, the representative contacts six potential clients. What is the probability that a sale will be made with (a) all six contacts, (b) none of the contacts, and (c) at least one contact?

58. ***A Boy or a Girl?*** Assume that the probability of the birth of a child of a particular sex is 50%. In a family with four children, what is the probability that (a) all the children are boys, (b) all the children are the same sex, and (c) there is at least one boy?

59. ***Estimating π*** A coin of diameter d is dropped onto a paper that contains a grid of squares d units on a side (see figure on the next page).

(a) Find the probability that the coin covers a vertex of one of the squares on the grid.

(b) Perform the experiment 100 times and use the results to approximate π.

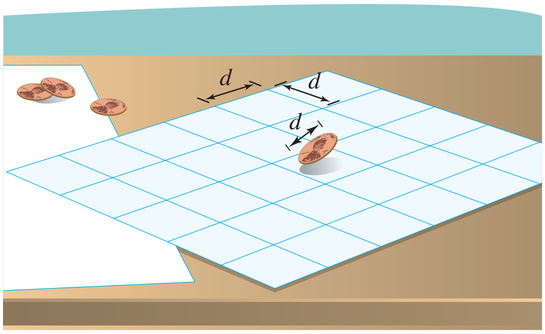
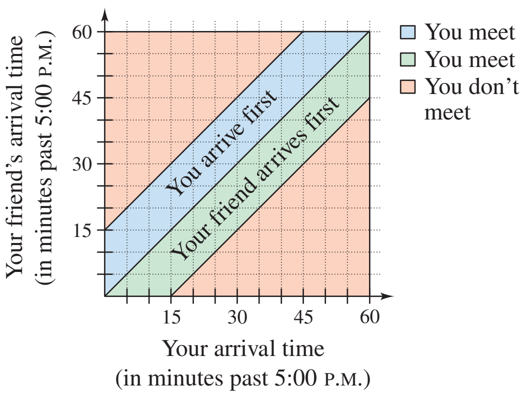


Figure for 59

60. ***Geometry*** You and a friend agree to meet at your favorite fast food restaurant between 5:00 and 6:00 P.M. The one who arrives first will wait 15 minutes for the other, after which the first person will leave (see figure). What is the probability that the two of you will actually meet, assuming that your arrival times are random within the hour?



Synthesis

True or False? **In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.**

61. If the probability of an outcome in a sample space is 1, then the probability of the other outcomes in the sample space is 0.

62. Rolling a number less than 3 on a normal six-sided die has a probability of $\frac{1}{3}$. The complement of this event is to roll a number greater than 3, and its probability is $\frac{1}{2}$.

63. ***Pattern Recognition and Exploration*** Consider a group of n people.

(a) Explain why the following pattern gives the probability that the n people have distinct birthdays.

$$n = 2: \quad \frac{365}{365} \cdot \frac{364}{365} = \frac{365 \cdot 364}{365^2}$$

$$n = 3: \quad \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} = \frac{365 \cdot 364 \cdot 363}{365^3}$$

(b) Use the pattern in part (a) to write an expression for the probability that four people ($n = 4$) have distinct birthdays.

(c) Let P_n be the probability that the n people have distinct birthdays. Verify that this probability can be obtained recursively by

$$P_1 = 1 \quad \text{and} \quad P_n = \frac{365 - (n - 1)}{365} P_{n-1}.$$

(d) Explain why $Q_n = 1 - P_n$ gives the probability that at least two people in a group of n people have the same birthday.

(e) Use the results of parts (c) and (d) to complete the table.

n	10	15	20	23	30	40	50
P_n							
Q_n							

(f) How many people must be in a group so that the probability of at least two of them having the same birthday is greater than $\frac{1}{2}$? Explain.

64. ***Think About It*** The weather forecast indicates that the probability of rain is 40%. Explain what this means.

Skills Review

In Exercises 65–68, solve the rational equation.

65. $\dfrac{2}{x - 5} = 4$

66. $\dfrac{3}{2x + 3} - 4 = \dfrac{-1}{2x + 3}$

67. $\dfrac{3}{x - 2} + \dfrac{x}{x + 2} = 1$

68. $\dfrac{2}{x} - \dfrac{5}{x - 2} = \dfrac{-13}{x^2 - 2x}$

In Exercises 69–72, solve the equation algebraically. Round your result to three decimal places.

69. $e^x + 7 = 35$

70. $200e^{-x} = 75$

71. $4 \ln 6x = 16$

72. $5 \ln 2x - 4 = 11$

In Exercises 73–76, evaluate $_nP_r$. Verify your result using a graphing utility.

73. $_5P_3$

74. $_{10}P_4$

75. $_{11}P_8$

76. $_9P_2$

In Exercises 77–80, evaluate $_nC_r$. Verify your result using a graphing utility.

77. $_6C_2$

78. $_9C_5$

79. $_{11}C_8$

80. $_{16}C_{13}$

What Did You Learn?

Key Terms

infinite sequence, *p. 580*
finite sequence, *p. 580*
recursively defined sequence, *p. 582*
n factorial, *p. 582*
summation notation, *p. 584*
series, *p. 585*
arithmetic sequence, *p. 592*
common difference, *p. 592*
geometric sequence, *p. 601*
common ratio, *p. 601*
first differences, *p. 616*
second differences, *p. 616*
binomial coefficients, *p. 619*
Pascal's Triangle, *p. 623*
Fundamental Counting Principle, *p. 628*
sample space, *p. 637*
mutually exclusive events, *p. 641*

Key Concepts

8.1 ■ Find the sum of an infinite sequence

Consider the infinite sequence $a_1, a_2, a_3, \dots a_i, \dots$.

1. The sum of the first n terms of the sequence is the finite series or partial sum of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

where i is the index of summation, n is the upper limit of summation, and 1 is the lower limit of summation.

2. The sum of all terms of the infinite sequence is called an infinite series and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

8.2 ■ Find the *n*th term and the *n*th partial sum of an arithmetic sequence

1. The nth term of an arithmetic sequence is $a_n = dn + c$, where d is the common difference between consecutive terms and $c = a_1 - d$.
2. The sum of a finite arithmetic sequence with n terms is given by $S_n = (n/2)(a_1 + a_n)$.

8.3 ■ Find the *n*th term and the *n*th partial sum of a geometric sequence

1. The nth term of a geometric sequence is $a_n = a_1 r^{n-1}$, where r is the common ratio of consecutive terms.
2. The sum of a finite geometric sequence $a_1, a_1 r, a_1 r^2, a_1 r^3, a_1 r^4, \dots, a_1 r^{n-1}$ with common ratio $r \neq 1$ is $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right)$.

8.3 ■ Find the sum of an infinite geometric series

If $|r| < 1$, then the infinite geometric series $a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + \cdots$ has the sum

$$S = \sum_{i=0}^{\infty} a_1 r^i = \frac{a_1}{1 - r}.$$

8.4 ■ Use mathematical induction

Let P_n be a statement with the positive integer n. If P_1 is true, and the truth of P_k implies the truth of P_{k+1} for every positive integer k, then P_n must be true for all positive integers n.

8.5 ■ Use the Binomial Theorem to expand a binomial

In the expansion of $(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$, the coefficient of $x^{n-r}y^r$ is ${}_nC_r = n!/[(n - r)!r!]$.

8.6 ■ Solve counting problems

1. If one event can occur in m_1 different ways and a second event can occur in m_2 different ways, then the number of ways that the two events can occur is $m_1 \cdot m_2$.
2. The number of permutations of n elements is $n!$.
3. The number of permutations of n elements taken r at a time is given by ${}_nP_r = n!/(n - r)!$
4. The number of distinguishable permutations of n objects is given by $n!/(n_1! \cdot n_2! \cdot n_3! \cdots n_k!)$.
5. The number of combinations of n elements taken r at a time is given by ${}_nC_r = n!/[(n - r)!r!]$.

8.7 ■ Find probabilities of events

1. If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, the probability of event E is $P(E) = n(E)/n(S)$.
2. If A and B are events in the same sample space, the probability of A or B occurring is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.
3. If A and B are independent events, the probability that A and B will occur is $P(A \text{ and } B) = P(A) \cdot P(B)$.
4. Let A be an event and let A' be its complement. If the probability of A is $P(A)$, then the probability of the complement is $P(A') = 1 - P(A)$.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

8.1 In Exercises 1–4, write the first five terms of the sequence. (Assume n begins with 1.)

1. $a_n = \dfrac{2^n}{2^n + 1}$ **2.** $a_n = \dfrac{1}{n} - \dfrac{1}{n+1}$

3. $a_n = \dfrac{(-1)^n}{n!}$ **4.** $a_n = \dfrac{(-1)^n}{(2n+1)!}$

In Exercises 5–8, write an expression for the *apparent* nth term of the sequence. (Assume n begins with 1.)

5. $5, 10, 15, 20, 25, \ldots$ **6.** $50, 48, 46, 44, 42, \ldots$

7. $2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \ldots$ **8.** $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \ldots$

In Exercises 9 and 10, write the first five terms of the sequence defined recursively.

9. $a_1 = 9, a_{k+1} = a_k - 4$ **10.** $a_1 = 49, a_{k+1} = a_k + 6$

In Exercises 11–14, simplify the factorial expression.

11. $\dfrac{18!}{20!}$ **12.** $\dfrac{10!}{8!}$

13. $\dfrac{(n+1)!}{(n-1)!}$ **14.** $\dfrac{2n!}{(n+1)!}$

In Exercises 15–22, find the sum.

15. $\displaystyle\sum_{i=1}^{6} 5$ **16.** $\displaystyle\sum_{k=2}^{5} 4k$

17. $\displaystyle\sum_{j=1}^{4} \frac{6}{j^2}$ **18.** $\displaystyle\sum_{i=1}^{8} \frac{i}{i+1}$

19. $\displaystyle\sum_{k=1}^{100} 2k^3$ **20.** $\displaystyle\sum_{j=0}^{40} (j^2 + 1)$

21. $\displaystyle\sum_{n=0}^{50} (n^2 + 3)$ **22.** $\displaystyle\sum_{n=1}^{100} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

In Exercises 23–26, use sigma notation to write the sum. Then use a graphing utility to find the sum.

23. $\dfrac{1}{2(1)} + \dfrac{1}{2(2)} + \dfrac{1}{2(3)} + \cdots + \dfrac{1}{2(20)}$

24. $2(1^2) + 2(2^2) + 2(3^2) + \cdots + 2(9^2)$

25. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \cdots + \dfrac{9}{10}$

26. $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \cdots$

In Exercises 27–30, find (a) the fourth partial sum and (b) the sum of the infinite series.

27. $\displaystyle\sum_{k=1}^{\infty} \frac{5}{10^k}$ **28.** $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2^k}$

29. $\displaystyle\sum_{k=1}^{\infty} 2(0.5)^k$ **30.** $\displaystyle\sum_{k=1}^{\infty} 4(0.25)^k$

31. ***Compound Interest*** A deposit of \$2500 is made in an account that earns 2% interest compounded quarterly. The balance in the account after n quarters is given by

$$a_n = 2500\left(1 + \frac{0.02}{4}\right)^n, \qquad n = 1, 2, 3, \ldots.$$

(a) Compute the first eight terms of this sequence.

(b) Find the balance in this account after 10 years by computing the 40th term of the sequence.

32. ***Education*** The numbers a_n of full-time faculty (in thousands) employed in institutions of higher education in the United States from 1991 to 2003 can be approximated by the model

$$a_n = 0.41n^2 + 2.7n + 532, \quad n = 1, 2, 3, \ldots, 13$$

where n is the year, with $n = 1$ corresponding to 1991. (Source: U.S. National Center for Education Statistics)

(a) Find the terms of this finite sequence for the given values of n.

(b) Use a graphing utility to graph the sequence for the given values of n.

(c) Use a graphing utility to construct a bar graph of the sequence for the given values of n.

(d) Use the sequence to predict the numbers of full-time faculty for the years 2004 to 2010. Do your results seem reasonable? Explain.

8.2 In Exercises 33–36, determine whether or not the sequence is arithmetic. If it is, find the common difference.

33. $5, 3, 1, -1, -3, \ldots$ **34.** $0, 1, 3, 6, 10, \ldots$

35. $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ **36.** $\frac{9}{9}, \frac{8}{9}, \frac{7}{9}, \frac{6}{9}, \frac{5}{9}, \ldots$

In Exercises 37–40, write the first five terms of the arithmetic sequence.

37. $a_1 = 3, \ d = 4$ **38.** $a_1 = 8, \ d = -2$

39. $a_4 = 10, \ a_{10} = 28$ **40.** $a_2 = 14, \ a_6 = 22$

In Exercises 41–44, write the first five terms of the arithmetic sequence. Find the common difference and write the nth term of the sequence as a function of n.

41. $a_1 = 35, \quad a_{k+1} = a_k - 3$

42. $a_1 = 15, \quad a_{k+1} = a_k + \frac{5}{2}$

43. $a_1 = 9, \quad a_{k+1} = a_k + 7$

44. $a_1 = 100, \quad a_{k+1} = a_k - 5$

In Exercises 45 and 46, find a formula for a_n for the arithmetic sequence and find the sum of the first 20 terms of the sequence.

45. $a_1 = 100, \quad d = -3$

46. $a_1 = 10, \quad a_3 = 28$

In Exercises 47–50, find the partial sum. Use a graphing utility to verify your result.

47. $\sum_{j=1}^{10} (2j - 3)$

48. $\sum_{j=1}^{8} (20 - 3j)$

49. $\sum_{k=1}^{11} \left(\frac{2}{3}k + 4\right)$

50. $\sum_{k=1}^{25} \left(\frac{3k + 1}{4}\right)$

51. Find the sum of the first 100 positive multiples of 5.

52. Find the sum of the integers from 20 to 80 (inclusive).

53. ***Job Offer*** The starting salary for an accountant is \$34,000 with a guaranteed salary increase of \$2250 per year for the first 4 years of employment. Determine (a) the salary during the fifth year and (b) the total compensation through 5 full years of employment.

54. ***Baling Hay*** In his first trip baling hay around a field, a farmer makes 123 bales. In his second trip he makes 11 fewer bales. Because each trip is shorter than the preceding trip, the farmer estimates that the same pattern will continue. Estimate the total number of bales made if there are another six trips around the field.

8.3 In Exercises 55–58, determine whether or not the sequence is geometric. If it is, find the common ratio.

55. 5, 10, 20, 40, . . .

56. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

57. 54, -18, 6, -2, . . .

58. $\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{8}{3}, \ldots$

In Exercises 59–62, write the first five terms of the geometric sequence.

59. $a_1 = 4, \; r = -\frac{1}{4}$

60. $a_1 = 2, \; r = \frac{3}{2}$

61. $a_1 = 9, \; a_3 = 4$

62. $a_1 = 2, \; a_3 = 12$

In Exercises 63–66, write the first five terms of the geometric sequence. Find the common ratio and write the nth term of the sequence as a function of n.

63. $a_1 = 120, \quad a_{k+1} = \frac{1}{3}a_k$

64. $a_1 = 200, \quad a_{k+1} = 0.1a_k$

65. $a_1 = 25, \quad a_{k+1} = -\frac{3}{5}a_k$

66. $a_1 = 18, \quad a_{k+1} = \frac{5}{3}a_k$

In Exercises 67–70, find the nth term of the geometric sequence and find the sum of the first 20 terms of the sequence.

67. $a_1 = 16, \quad a_2 = -8$

68. $a_3 = 6, \quad a_4 = 1$

69. $a_1 = 100, \quad r = 1.05$

70. $a_1 = 5, \quad r = 0.2$

In Exercises 71–78, find the sum. Use a graphing utility to verify your result.

71. $\sum_{i=1}^{7} 2^{i-1}$

72. $\sum_{i=1}^{5} 3^{i-1}$

73. $\sum_{n=1}^{7} (-4)^{n-1}$

74. $\sum_{n=1}^{4} 12\left(-\frac{1}{2}\right)^{n-1}$

75. $\sum_{n=0}^{4} 250(1.02)^n$

76. $\sum_{n=0}^{5} 400(1.08)^n$

77. $\sum_{i=1}^{10} 10\left(\frac{3}{5}\right)^{i-1}$

78. $\sum_{i=1}^{15} 20(0.2)^{i-1}$

In Exercises 79–82, find the sum of the infinite geometric series.

79. $\sum_{i=1}^{\infty} 4\left(\frac{7}{8}\right)^{i-1}$

80. $\sum_{i=1}^{\infty} 6\left(\frac{1}{3}\right)^{i-1}$

81. $\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^{k-1}$

82. $\sum_{k=1}^{\infty} 1.3\left(\frac{1}{10}\right)^{k-1}$

83. ***Depreciation*** A company buys a fleet of six vans for \$120,000. During the next 5 years, the fleet will depreciate at a rate of 30% per year. (That is, at the end of each year, the depreciated value will be 70% of the value at the beginning of the year.)

(a) Find the formula for the nth term of a geometric sequence that gives the value of the fleet t full years after it was purchased.

(b) Find the depreciated value of the fleet at the end of 5 full years.

84. ***Annuity*** A deposit of \$75 is made at the beginning of each month in an account that pays 4% interest, compounded monthly. The balance A in the account at the end of 4 years is given by

$$A = 75\left(1 + \frac{0.04}{12}\right)^1 + \cdots + 75\left(1 + \frac{0.04}{12}\right)^{48}.$$

Find A.

8.4 In Exercises 85–88, use mathematical induction to prove the formula for every positive integer n.

85. $2 + 7 + \cdots + (5n - 3) = \frac{n}{2}(5n - 1)$

86. $1 + \frac{3}{2} + 2 + \frac{5}{2} + \cdots + \frac{1}{2}(n + 1) = \frac{n}{4}(n + 3)$

87. $\sum_{i=0}^{n-1} ar^i = \frac{a(1 - r^n)}{1 - r}$

88. $\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2}[2a + (n - 1)d]$

In Exercises 89–92, find the sum using the formulas for the sums of powers of integers.

89. $\sum_{n=1}^{30} n$

90. $\sum_{n=1}^{10} n^2$

91. $\sum_{n=1}^{7} (n^4 - n)$

92. $\sum_{n=1}^{6} (n^5 - n^2)$

In Exercises 93–96, write the first five terms of the sequence beginning with a_1. Then calculate the first and second differences of the sequence. Does the sequence have a linear model, a quadratic model, or neither?

93. $a_1 = 5$
$a_n = a_{n-1} + 5$

94. $a_1 = -3$
$a_n = a_{n-1} - 2n$

95. $a_1 = 16$
$a_n = a_{n-1} - 1$

96. $a_1 = 1$
$a_n = n - a_{n-1}$

8.5 In Exercises 97–100, find the binomial coefficient. Use a graphing utility to verify your result.

97. ${}_{10}C_8$

98. ${}_{12}C_5$

99. $\binom{9}{4}$

100. $\binom{14}{12}$

In Exercises 101–104, use Pascal's Triangle to find the binomial coefficient.

101. ${}_{6}C_3$

102. ${}_{9}C_7$

103. $\binom{8}{4}$

104. $\binom{10}{5}$

In Exercises 105–110, use the Binomial Theorem to expand and simplify the expression. (Recall that $i = \sqrt{-1}$.)

105. $(x + 5)^4$

106. $(y - 3)^3$

107. $(a - 4b)^5$

108. $(3x + y)^7$

109. $(7 + 2i)^4$

110. $(4 - 5i)^3$

8.6

111. ***Numbers in a Hat*** Slips of paper numbered 1 through 14 are placed in a hat. In how many ways can two numbers be drawn so that the sum of the numbers is 12? Assume the random selection is without replacement.

112. ***Aircraft Boarding*** Eight people are boarding an aircraft. Two have tickets for first class and board before those in economy class. In how many ways can the eight people board the aircraft?

113. ***Course Schedule*** A college student is preparing a course schedule of four classes for the next semester. The student can choose from the open sections shown in the table.

Course	Sections
Math 100	001–004
Economics 110	001–003
English 105	001–006
Humanities 101	001–003

(a) Find the number of possible schedules that the student can create from the offerings.

(b) Find the number of possible schedules that the student can create from the offerings if two of the Math 100 sections are closed.

(c) Find the number of possible schedules that the student can create from the offerings if two of the Math 100 sections and four of the English 105 sections are closed.

114. ***Telemarketing*** A telemarketing firm is making calls to prospective customers by randomly dialing a seven-digit phone number within an area code.

(a) Find the number of possible calls that the telemarketer can make.

(b) If the telemarketing firm is calling only within an exchange that begins with a "7" or a "6", how many different calls are possible?

(c) If the telemarketing firm is calling only within an exchange that does not begin with a "0" or a "1," how many calls are possible?

In Exercises 115–122, evaluate the expression. Use a graphing utility to verify your result.

115. ${}_{10}C_8$

116. ${}_{8}C_6$

117. ${}_{12}P_{10}$

118. ${}_{6}P_4$

119. ${}_{100}C_{98}$

120. ${}_{50}C_{48}$

121. ${}_{1000}P_2$

122. ${}_{500}P_2$

In Exercises 123 and 124, find the number of distinguishable permutations of the group of letters.

123. C, A, L, C, U, L, U, S

124. I, N, T, E, G, R, A, T, E

125. ***Sports*** There are 10 bicyclists entered in a race. In how many different orders could the 10 bicyclists finish? (Assume there are no ties.)

126. ***Sports*** From a pool of seven juniors and eleven seniors, four co-captains will be chosen for the football team. How many different combinations are possible if two juniors and two seniors are to be chosen?

127. ***Exam Questions*** A student can answer any 15 questions from a total of 20 questions on an exam. In how many different ways can the student select the questions?

128. ***Lottery*** In the Lotto Texas game, a player chooses six distinct numbers from 1 to 54. In how many ways can a player select the six numbers?

In Exercises 129 and 130, solve for *n*.

129. ${}_{n+1}P_2 = 4 \cdot {}_nP_1$ **130.** $8 \cdot {}_nP_2 = {}_{n+1}P_3$

8.7

131. ***Apparel*** A man has five pairs of socks (no two pairs are the same color). He randomly selects two socks from a drawer. What is the probability that he gets a matched pair?

132. ***Bookshelf Order*** A child returns a five-volume set of books to a bookshelf. The child is not able to read, and so cannot distinguish one volume from another. What is the probability that the books are shelved in the correct order?

133. ***Data Analysis*** A sample of college students, faculty members, and administrators were asked whether they favored a proposed increase in the annual activity fee to enhance student life on campus. The results of the study are shown in the table.

	Favor	Oppose	Total
Students	237	163	400
Faculty	37	38	75
Admin.	18	7	25
Total	292	208	500

A person is selected at random from the sample. Find each probability.

(a) The person is not in favor of the proposal.

(b) The person is a student.

(c) The person is a faculty member and is in favor of the proposal.

134. ***Tossing a Die*** A six-sided die is rolled six times. What is the probability that each side appears exactly once?

135. ***Poker Hand*** Five cards are drawn from an ordinary deck of 52 playing cards. Find the probability of getting two pairs. (For example, the hand could be A-A-5-5-Q or 4-4-7-7-K.)

136. ***Drawing a Card*** You randomly select a card from a 52-card deck. What is the probability that the card is *not* a club?

Synthesis

True or False? **In Exercises 137 and 138, determine whether the statement is true or false. Justify your answer.**

137. $\dfrac{(n+2)!}{n!} = (n+2)(n+1)$ **138.** $\displaystyle\sum_{k=1}^{8} 3k = 3\sum_{k=1}^{8} k$

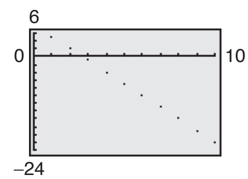
139. ***Writing*** In your own words, explain what makes a sequence (a) arithmetic and (b) geometric.

140. ***Think About It*** How do the two sequences differ?

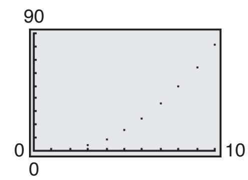
(a) $a_n = \dfrac{(-1)^n}{n}$ (b) $a_n = \dfrac{(-1)^{n+1}}{n}$

141. ***Graphical Reasoning*** The graphs of two sequences are shown below. Identify each sequence as arithmetic or geometric. Explain your reasoning.

(a)



(b)



142. ***Population Growth*** Consider an idealized population with the characteristic that each member of the population produces one offspring at the end of every time period. If each member has a life span of three time periods and the population begins with 10 newborn members, then the following table shows the populations during the first five time periods.

Age Bracket	Time Period				
	1	**2**	**3**	**4**	**5**
0–1	10	10	20	40	70
1–2		10	10	20	40
2–3			10	10	20
Total	10	20	40	70	130

The sequence for the total populations has the property that

$S_n = S_{n-1} + S_{n-2} + S_{n-3}, \quad n > 3.$

Find the total populations during the next five time periods.

143. ***Writing*** Explain what a recursion formula is.

144. ***Writing*** Explain why the terms of a geometric sequence of positive terms decrease when $0 < r < 1$.

145. ***Think About It*** How do the expansions of $(x - y)^n$ and $(-x + y)^n$ differ?

146. The probability of an event must be a real number in what interval? Is the interval open or closed?

8 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–4, write the first five terms of the sequence.

1. $a_n = \left(-\frac{2}{3}\right)^{n-1}$ (Begin with $n = 1$.) **2.** $a_1 = 12$ and $a_{k+1} = a_k + 4$

3. $b_n = \dfrac{(-1)^n x^n}{n}$ (Begin with $n = 1$.) **4.** $b_n = \dfrac{(-1)^{2n+1} x^{2n+1}}{(2n+1)!}$ (Begin with $n = 1$.)

5. Simplify $\dfrac{11! \cdot 4!}{4! \cdot 7!}$. **6.** Simplify $\dfrac{n!}{(n+1)!}$. **7.** Simplify $\dfrac{2n!}{(n-1)!}$.

8. Write an expression for the *apparent* nth term of the sequence 2, 5, 10, 17, 26, ... (Assume n begins with 1).

In Exercises 9 and 10, find a formula for the *n*th term of the sequence.

9. Arithmetic: $a_1 = 5000$, $d = -100$ **10.** Geometric: $a_1 = 4$, $a_{k+1} = \frac{1}{2}a_k$

11. Use sigma notation to write $\dfrac{2}{3(1) + 1} + \dfrac{2}{3(2) + 1} + \cdots + \dfrac{2}{3(12) + 1}$.

12. Use sigma notation to write $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots$.

In Exercises 13–15, find the sum.

13. $\displaystyle\sum_{n=1}^{7} (8n - 5)$ **14.** $\displaystyle\sum_{n=1}^{8} 24\left(\frac{1}{6}\right)^{n-1}$ **15.** $5 - 2 + \frac{4}{5} - \frac{8}{25} + \frac{16}{125} - \cdots$

16. Use mathematical induction to prove the formula

$$3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2}.$$

17. Use the Binomial Theorem to expand and simplify $(2a - 5b)^4$.

In Exercises 18–21, evaluate the expression.

18. ${}_9C_3$ **19.** ${}_{20}C_3$ **20.** ${}_9P_2$ **21.** ${}_{70}P_3$

22. Solve for n in $4 \cdot {}_nP_3 = {}_{n+1}P_4$.

23. How many distinct license plates can be issued consisting of one letter followed by a three-digit number?

24. Four students are randomly selected from a class of 25 to answer questions from a reading assignment. In how many ways can the four be selected?

25. A card is drawn from a standard deck of 52 playing cards. Find the probability that it is a red face card.

26. In 2006, six of the eleven men's basketball teams in the Big Ten Conference were to participate in the NCAA Men's Basketball Championship Tournament. If six of the eleven schools are selected at random, what is the probability that the six teams chosen were the actual six teams selected to play?

27. Two integers from 1 to 60 are chosen by a random number generator. What is the probability that (a) both numbers are odd, (b) both numbers are less than 12, and (c) the same number is chosen twice?

28. A weather forecast indicates that the probability of snow is 75%. What is the probability that it will not snow?

Proofs in Mathematics

Properties of Sums (p. 585)

1. $\sum_{i=1}^{n} c = cn$, c is a constant.

2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, c is a constant.

3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Proof

Each of these properties follows directly from the properties of real numbers.

1. $\sum_{i=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{n \text{ terms}} = cn$

The Distributive Property is used in the proof of Property 2.

2. $$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n$$
$$= c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i$$

The proof of Property 3 uses the Commutative and Associative Properties of Addition.

3.
$$\sum_{i=1}^{n} (a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n)$$
$$= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)$$
$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

The proof of Property 4 uses the Commutative and Associative Properties of Addition and the Distributive Property.

4.
$$\sum_{i=1}^{n} (a_i - b_i) = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + \cdots + (a_n - b_n)$$
$$= (a_1 + a_2 + a_3 + \cdots + a_n) + (-b_1 - b_2 - b_3 - \cdots - b_n)$$
$$= (a_1 + a_2 + a_3 + \cdots + a_n) - (b_1 + b_2 + b_3 + \cdots + b_n)$$
$$= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Infinite Series

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity; throughout the next quarter of the interval at double the intensity; throughout the following eighth at triple the intensity and so ad infinitum; The average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots$$

is 2.

The Sum of a Finite Arithmetic Sequence (p. 595)

The sum of a finite arithmetic sequence with n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Proof

Begin by generating the terms of the arithmetic sequence in two ways. In the first way, repeatedly add d to the first term to obtain

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n \\ &= a_1 + [a_1 + d] + [a_1 + 2d] + \cdots + [a_1 + (n-1)d]. \end{aligned}$$

In the second way, repeatedly subtract d from the nth term to obtain

$$\begin{aligned} S_n &= a_n + a_{n-1} + a_{n-2} + \cdots + a_3 + a_2 + a_1 \\ &= a_n + [a_n - d] + [a_n - 2d] + \cdots + [a_n - (n-1)d]. \end{aligned}$$

If you add these two versions of S_n, the multiples of d subtract out and you obtain

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) \quad n \text{ terms}$$

$$2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n}{2}(a_1 + a_n).$$

The Sum of a Finite Geometric Sequence (p. 604)

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ \ldots,\ a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \displaystyle\sum_{i=1}^{n} a_1r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

Proof

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}$$

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + a_1r^n \quad \text{Multiply by } r.$$

Subtracting the second equation from the first yields

$$S_n - rS_n = a_1 - a_1r^n.$$

So, $S_n(1 - r) = a_1(1 - r^n)$, and, because $r \neq 1$, you have $S_n = a_1\left(\dfrac{1-r^n}{1-r}\right)$.

The Binomial Theorem (p. 619)

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdot \cdot \cdot + {}_nC_r\, x^{n-r}y^r + \cdot \cdot \cdot + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

Proof

The Binomial Theorem can be proved quite nicely using mathematical induction. The steps are straightforward but look a little messy, so only an outline of the proof is presented.

1. If $n = 1$, you have $(x + y)^1 = x^1 + y^1 = {}_1C_0x + {}_1C_1y$, and the formula is valid.

2. Assuming that the formula is true for $n = k$, the coefficient of $x^{k-r}y^r$ is

$${}_kC_r = \frac{k!}{(k-r)!r!} = \frac{k(k-1)(k-2) \cdot \cdot \cdot (k-r+1)}{r!}.$$

To show that the formula is true for $n = k + 1$, look at the coefficient of $x^{k+1-r}y^r$ in the expansion of

$$(x + y)^{k+1} = (x + y)^k(x + y).$$

From the right-hand side, you can determine that the term involving $x^{k+1-r}y^r$ is the sum of two products.

$$\begin{aligned}
&({}_kC_r x^{k-r}y^r)(x) + ({}_kC_{r-1}x^{k+1-r}y^{r-1})(y)\\
&= \left[\frac{k!}{(k-r)!r!} + \frac{k!}{(k+1-r)!(r-1)!}\right]x^{k+1-r}y^r\\
&= \left[\frac{(k+1-r)k!}{(k+1-r)!r!} + \frac{k!r}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= \left[\frac{k!(k+1-r+r)}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= \left[\frac{(k+1)!}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= {}_{k+1}C_r\, x^{k+1-r}y^r
\end{aligned}$$

So, by mathematical induction, the Binomial Theorem is valid for all positive integers n.